The Dissertation Committee for Inderjeet Singh
certifies that this is the approved version of the following dissertation:

Risk-Averse Periodic Preventive Maintenance Optimization

Committee:

Elmira Popova, Supervisor

David P. Morton, Co-Supervisor

Paul Damien

John J. Hasenbein

Erhan Kutanoglu

Risk-Averse Periodic Preventive Maintenance Optimization

by

Inderjeet Singh, M.Tech.; B.Tech.

Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

August 2011

Dedicated to my parents and my late grandparents.

Acknowledgments

I wonder if I could have finished this dissertation without the guidance, support, encouragement and blessings of many people around me. Therefore, I would like to take this opportunity to thank them all, for making this real.

First and foremost, I thank my advisor Dr. Elmira Popova and my co-advisor Dr. David P. Morton for accepting me as their doctoral student. I have utmost respect for their vast knowledge, experience and patience. Their influence on me as a person and as a scholar will go way past this dissertation.

Dr. Popova, you have been a constant source of guidance, support and motivation during every stage of my doctoral research. I am grateful to you for your valuable advice and tutelage. I thank you for providing financial support, and for giving me the freedom to work independently while ensuring that I stayed on the right track. I also thank you for providing guidance on other walks of my life. I can only wish to be as good as you.

Dr. Morton, you are the best teacher I have had in my academic life. I have always enjoyed your courses. I thank you for motivating me to think more deeply about my research. I am also grateful to you for meticulously proof reading my dissertation and for your invaluable feedback. You have substantially improved my writing style. Your approval means a lot to me.

I thank Dr. Paul Damien, Dr. John J. Hasenbein and Dr. Erhan Kutanoglu for agreeing to serve on my doctoral dissertation committee, and for reviewing my research. Dr. Damien, your guest lectures in ORIE inspired me to work on the Bayesian methods. I wish you had offered a graduate level course in Bayesian statistics during my graduate studies. Dr. Hasenbein, thank you for stimulating my interest in the stochastic area. Before joining the ORIE program, I always felt *queasy* in stochastic courses. But your first year course on applied probability is the

reason that I am able to do research in this area. Dr. K, thank you for keeping the IE flag fluttering high in the ORIE program. I really enjoyed your production and inventory control class.

I thank Dr. Nedialko B. Dimitrov for providing valuable guidelines to improve writing and presentation skills. I have tried my best to keep up with these guidelines throughout this dissertation and in all my presentations so far.

I thank Diana Ziegler for the wonderful administrative support she extended throughout my graduate studies in ORIE.

A very important component of my research came from my interaction with the risk management group at the South Texas Project Nuclear Operating Company (STPNOC). I thank Dr. Popova and Ernie Kee for giving me the opportunity to work as a summer intern at STPNOC. This internship helped me to validate key assumptions used in my research and to gain insights on the applicability of my work in a real nuclear power plant. I also thank Dr. Teresa E. Tutt for giving me the opportunity to collaborate with her on the development of preventive maintenance optimization software.

Dr. Bugao Xu, thank you for giving me the opportunity to teach your lab course, and for providing financial support when I needed it the most. I thank you for your confidence in my abilities as a teacher. I really enjoyed teaching the course. Dr. Mourad Krifa, thank you for appointing me as your research assistant for a summer semester. I am glad that I could make a small contribution to your research.

I learned computational Bayesian methods using the LearnBayes package. Dr. Jim Albert, thank you for maintaining and updating this package on the Comprehensive R Archive Network. At the same time, I am grateful to Thomas Kirschenmann for introducing this package to me and for sharing his sample R codes. I also thank Thomas for his kind help during my practice talk for my final oral exam.

I thank Dr. Nat Jack for providing the basis for an important part of my research. I am grateful to Dr. Rose Baker for providing the complete syringe-driver infusion pump dataset that kick-started my computational results. I thank you for taking your valuable time to dig out a twenty years old dataset that Dr. Nat Jack was unable to locate. It re-affirms my belief that women have better house keeping skills than men.

I gained valuable industry experience during two summer internships at Dell Inc. For this I thank my friend Samer Awwad for recommending me for this position. I also thank my friends Shamsher Singh, Saurabh Bansal, and Kanthimathi Sathasivan for their help and support at various stages of my PhD. I am indebted to the Haar family from Austin, TX for their warmth and kind help.

I thank my parents-in-law, Chanchal Singh and Swarnjit Kaur, for having faith in a poor graduate student while giving their daughter's hand in marriage. I also thank Keerat, Sukhbir and Jaspreet for their support.

I thank my family for their unconditional love and encouragement throughout my life. I am here because of my elder sister, Ravinder Kaur, who developed my interest in science and math subjects. I am grateful to my brother-in-law, Prof. Sukhwinderpal Singh for developing my interest in teaching and research. I thank my elder brother, Paramjit Singh, and my sister-in-law, Amita, for always keeping a check on me. The little imps in my family; Harleen, Gurleen, Gurnoor, Banipreet and Jaisleen always kept me excited throughout my PhD.

I thank my wife, Sukhdeep Kaur for her endless love and affection, and for always standing by me in the lowest moments of my life.

I thank my parents, Gurdeep Singh and Satnam Kaur, and my *guruji*, Rajdeep Anand, for their blessings. They have always expressed how proud they are of me and how much they love me. I too am proud of them and love them very much. Last but not the least, I thank you *Babaji* for giving me such a wonderful life.

Risk-Averse Periodic Preventive Maintenance Optimization

Inderjeet Singh, Ph.D.
The University of Texas at Austin, 2011

Supervisor: Elmira Popova
Co-Supervisor: David P. Morton

We consider a class of periodic preventive maintenance (PM) optimization problems, for a single piece of equipment that deteriorates with time or use, and can be repaired upon failure, through corrective maintenance (CM). We develop analytical and simulation-based optimization models that seek an optimal periodic PM policy, which minimizes the sum of the expected total cost of PMs and the risk-averse cost of CMs, over a finite planning horizon. In the simulation-based models, we assume that both types of maintenance actions are imperfect, whereas our analytical models consider imperfect PMs with minimal CMs. The effectiveness of maintenance actions is modeled using age reduction factors. For a repairable unit of equipment, its virtual age, and not its calendar age, determines the associated failure rate. Therefore, two sets of parameters, one describing the effectiveness of maintenance actions, and the other that defines the underlying failure rate of a piece of equipment, are critical to our models. Under a given maintenance policy, the two sets of parameters and a virtual-age-based age-reduction model, completely define the failure process of a piece of equipment. In practice, the true failure rate, and exact quality of the maintenance actions, cannot be determined, and are often

estimated from the equipment failure history.

We use a Bayesian approach to parameter estimation, under which a random-walk-based Gibbs sampler provides posterior estimates for the parameters of interest. Our posterior estimates for a few datasets from the literature, are consistent with published results. Furthermore, our computational results successfully demonstrate that our Gibbs sampler is arguably the obvious choice over a general rejection sampling-based parameter estimation method, for this class of problems. We present a general simulation-based periodic PM optimization model, which uses the posterior estimates to simulate the number of operational equipment failures, under a given periodic PM policy. Optimal periodic PM policies, under the classical maximum likelihood (ML) and Bayesian estimates are obtained for a few datasets. Limitations of the ML approach are revealed for a dataset from the literature, in which the use of ML estimates of the parameters, in the maintenance optimization model, fails to capture a trivial optimal PM policy.

Finally, we introduce a single-stage and a two-stage formulation of the risk-averse periodic PM optimization model, with imperfect PMs and minimal CMs. Such models apply to a class of complex equipment with many parts, operational failures of which are addressed by replacing or repairing a few parts, thereby not affecting the failure rate of the equipment under consideration. For general values of PM age reduction factors, we provide sufficient conditions to establish the convexity of the first and second moments of the number of failures, and the risk-averse expected total maintenance cost, over a finite planning horizon. For increasing Weibull rates and a general class of increasing and convex failure rates, we show that these convexity results are independent of the PM age reduction factors. In general, the optimal periodic PM policy under the single-stage model is no better than the optimal two-stage policy. But if PMs are assumed perfect, then we establish that the single-stage and the two-stage optimization models are equivalent.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Fail proof operation of production systems has been always desired to meet targeted productivity goals. System availability, calendar-based production runs, fewer system disruptions, safe and reliable operation of the system, etc., are common examples of such productivity goals. Most production systems have multiple types of equipment which deteriorate with age and usage, and have limited useful life. Most equipment can be repaired upon failure. Therefore, the useful operational life of a production system of repairable equipment can be increased through proper maintenance.

Appropriate maintenance scheduling is also important for certain repairable systems, for which, operational failures can be critical. Nuclear power reactors, nuclear submarines, life support devices, and aircraft engines are common examples of such systems. On the other hand, maintenance activities generally require partial or complete shutdown of the system resulting in production losses. In addition, there are costs associated with maintenance and often the maintenance budget is limited. As a consequence, system managers face an important problem of balancing the maintenance costs and production losses against targeted productivity goals (Rausand & Høyland 2004, Popova et al. 2005).

1.1 Motivation

For nuclear power plants (NPPs), operational failures can be fatal. Therefore, such systems are designed to have high reliability. But operational failures of nuclear power plants at Chernobyl in the Ukraine[1] and Three Mile Island (TMI) in the USA are well known. The Chernobyl disaster, classified as a level 7 (highest) on the international nuclear and radiological event scale (INES), had widespread health and environment effects. On the other hand the TMI accident, a level 5 on INES, resulted in severe damage to the reactor core without any direct impact on people and the environment. Post TMI, the Nuclear Regulatory Commission (NRC) has enforced strict regulations to ensure safe operation of nuclear power plants in the USA. As a result there has been a substantial increase in construction costs and operating expenses associated with NPPs. Despite the reduction in profits, more than 100 nuclear power plants continue to operate safely in the USA. In 2007, the NRC received the first application in the last 30 years to build a new nuclear power plant in the USA. Clearly, nuclear energy would be more popular today had the Chernobyl and the TMI events never happened.

The nuclear industry now has the capability of advanced reactor designs that can be ordered off the shelf. The NRC has approved several advanced reactor designs that are more economical to build and even safer to operate. Due to rising demand for cleaner energy, nuclear power plants have gained popularity once again in recent times. As a result, the NRC has received 20 applications from various companies to build 31 new nuclear reactors in the USA since 2007 (see NRC 2011). The global picture of the revamping nuclear industry is similar. Currently, 124 NPPs have been planned and an additional 276 have been proposed for 43 countries worldwide, excluding the USA (see WNA 2011). However, it is still unclear that what will be the immediate effect of the ongoing nuclear crisis in Japan's Fukushima Daiichi nuclear power plants, on the emerging interest in nuclear power. But is it quite certain that diminishing supply of fossil fuels, along with political uncertainties

[1]formerly the Ukrainian Soviet Socialist Republic

regarding availability of existing fuel supplies, will ultimately lead to a global interest in the nuclear energy (McGoldrick 2011).

The maintenance models addressed in the current research draw motivation from the challenges faced by the South Texas Project Nuclear Operating Company (STPNOC) in scheduling maintenance activities for their NPPs. South Texas Project (STP), located in Bay City, TX, is one of the newest and the largest nuclear power plants in the USA and an industry leader in safety, reliability and efficiency. STP has two nuclear reactors that together can produce 2500 megawatts of electric power. The first reactor became operational in August 1988 and the second in June 1989; the former is the sixth and latter is the fourth youngest nuclear reactor of the 104 licensed nuclear power plants in the USA. These two reactors, have a license to operate for another 19 and 20 years respectively, therefore all the decisions taken by the board of directors have finite time horizons. Further, STP is currently building two new nuclear reactors and the risk management group at STP has already started working on the maintenance plans for the new reactors.

1.2 Problem Description

STP in particular, and the US nuclear industry in general, have a very strong culture of safety that prefers safety over production. Safe and reliable operation of any NPP relies on thousands of different types of equipment, each of which has a limited useful life. We use the term *equipment* to represent a single piece of equipment or a subsystem of different types of equipment. This terminology is in accordance with the level of detail considered in the failure analysis and maintenance planning at STP, and it is widely applicable to many other production systems.

Now, consider a unit of repairable equipment which fails unexpectedly while in service, and it is repaired upon failure during its useful life, before it is finally replaced. We refer to such repair actions as corrective maintenances (CMs). Typically, equipment can fail in different failure modes and depending upon the severity of the failure mode, there is a cost (possibly different) associated with each equipment failure-mode combination. Besides CMs, equipment may undergo scheduled maintenance from time to time, and here, the primary objective is to reduce the probability of operational (in service) equipment failures. Such scheduled maintenance actions are planned in advance and commonly known as preventive maintenances (PMs). In general, equipment can also have different types of PMs, each designed to mitigate a set of failure modes. Similar to CMs, there are costs associated with PMs. Typical components of CM and PM costs are direct labor and parts costs, and possible production losses due to equipment unavailability. In addition, there may be a third cost component to account for the risk associated with the maintenance actions.

We focus on the effectiveness of maintenance actions because they play a very important role in maintenance planning. The quality, or effectiveness, of a maintenance action is often modeled as perfect, minimal and imperfect (Pham & Wang 1996). It is assumed that perfect maintenance restores the equipment to *as good as new* condition, whereas minimal repair brings the equipment to *as good as old* condition. In a typical setting, imperfect maintenance brings the equipment to a state between *as good as old* and *as good as new* but in general, imperfect maintenance can yield a state that is worse than minimal repair. In this research, we focus our attention on the former case and regard minimal repair and perfect maintenance as the two extreme cases of imperfect maintenance.

In NPPs, estimation of maintenance effectiveness in the face of, either a very small failure dataset or heavily censored observations, can be challenging. In such situations, it makes sense to use prior knowledge of systems engineers, who are directly responsible for planning the maintenance activities, and to combine this information with existing equipment failure data to estimate the effectiveness of maintenance actions.

In this research, we consider finite horizon *periodic maintenance policy* due to its wide applicability in the nuclear industry. Under this policy, PMs are performed at times jT, $j = 1, 2, \ldots, n$, over a finite planning horizon L, and all intervening operational failures of the equipment are repaired through CMs. The optimal policy aims to minimize the expected value of a random cost function $\mathcal{C}(T)$ which is primarily driven by the total expected cost of CMs and PMs.

1.3 Notion of Failure Rate

In the reliability literature, maintenance problems are generally formulated in terms of a counting process $\{\mathcal{N}(t), t \geq 0\}$, where random variable $\mathcal{N}(t)$ denotes the number of equipment failures in the time interval $(0, t]$. In addition, it is assumed that the equipment is put into operation at time $t = 0$ and $\mathcal{N}(0) = 0$. Let $\Lambda(t) = \mathbb{E}\left[\mathcal{N}(t)\right]$ be the expected number of failures in time interval $(0, t]$, then *rate of occurrence of failures* (ROCOF), or simply the failure rate, $\lambda(t)$ of the counting process is given as follows (Rausand & Høyland 2004):

$$\lambda(t) = \Lambda'(t) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}\left[\mathcal{N}(t + \Delta t) - \mathcal{N}(t)\right]}{\Delta t}.$$

We assume that the counting process $\{\mathcal{N}(t), t \geq 0\}$ is regular i.e., for sufficiently small Δt

$$\mathcal{N}(t+\Delta t) - \mathcal{N}(t) = 0 \text{ or } 1.$$

Therefore,

$$\lambda(t) = \Lambda'(t) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}\left\{\mathcal{N}(t+\Delta t) - \mathcal{N}(t) = 1\right\}}{\Delta t}.$$

Since we focus on maintenance of repairable equipment, we find it important to differentiate between the *failure rate of a process* and the *failure rate of the lifetime distribution*. We use the former for repairable equipment and the latter applies to non-repairable equipment. The counting process associated with non-repairable equipment is fairly straightforward, in which $\mathcal{N}(t)$ takes on only two values, 0 or 1. Thus, for non-repairable equipment, $\Lambda(t) = \mathbb{E}\left[\mathcal{N}(t)\right] = F_X(t)$ and $\lambda(t) = f_X(t)$, where $F_X(.)$ and $f_X(.)$ are the distribution and density functions of the equipment's lifetime, X, respectively. Now consider the following function:

$$r(t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}\left\{t \leq X < t+\Delta t \mid X \geq t\right\}}{\Delta t},$$

and note that

$$r(t) = \frac{f_X(t)}{1 - F_X(t)} = -\frac{d}{dt}\ln(1 - F_X(t)).$$

In the reliability context, we call $r(t)$ the failure rate function of the lifetime distribution (not of the failure process) of non-repairable equipment. Since $r(t) \neq \lambda(t)$, we note that referring to a failure rate without stating whether it pertains to a process or a distribution can be ambiguous (Thompson 1981).

For equipment that undergo maintenance, it is possible to accommodate the history of the failure process and define the *conditional* ROCOF, or simply the conditional failure rate function, of the regular counting process as:

$$\lambda_c(t \mid \mathcal{H}_{t^-}) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}\{\mathcal{N}(t+\Delta t) - \mathcal{N}(t) = 1 \,|\mathcal{H}_{t^-}\}}{\Delta t},$$

where $\mathcal{H}_{t^-}$ denotes the history of the failure process up to (but not including) time t (Rausand & Høyland 2004).

1.4 Literature Review

The literature on preventive maintenance (PM) problems is enormous and can be classified in many different ways. For our purposes we restrict attention to maintenance models for a single unit of equipment and a system of multiple types of equipment.

1.4.1 Single Equipment Maintenance Models

We begin with a review of relevant single equipment PM models and refer the reader to excellent survey articles by McCall (1965), Pierskalla & Voelker (1979), Valdez-Flores & Feldman (1989) for additional work in the area. The reader may also refer to Rausand & Høyland (2004), a contemporary textbook on reliability.

Barlow & Hunter (1960) use renewal theory to characterize analytical solutions for optimal replacement times that maximize the long-run availability of (i) non-repairable equipment under the age replacement policy and (ii) a repairable complex system, which is minimally repaired upon failure under the periodic replacement policy. The equivalence between maximizing long-run availability and minimizing expected cost per unit time is established for both the policies, when expected repair and replacement times are substituted with expected repair and replacement costs, respectively. For the minimal repair periodic replacement policy it is shown that if a complex system is minimally repaired upon failure then the counting process $\{\mathcal{N}(t), 0 \leq t < \infty\}$ that records the number of failures (or equivalently number of minimal repairs) after t units of operating time since the last replacement, is a nonhomogeneous Poisson process (NHPP) with parameter $\lambda(t)$, where $\lambda(t)$ is the ROCOF.

The minimal repair periodic replacement policy of Barlow & Hunter (1960) has been extended by many researchers. Muth (1977) presents a modification in which the system is replaced when it fails for the first time after reaching the optimal replacement age. Park (1979) optimizes the minimal repair periodic replacement policy of Barlow & Hunter (1960) by changing the decision rule for the system replacement from clock based operating time T to number of minimal repairs N. Under the new policy, the system is replaced at the N^{th} failure. The expected cost per replacement cycle under the new policy is lower than that of the periodic replacement policy in Barlow & Hunter (1960). Nakagawa (1979) provides an extension (see the paper's model B) in which a PM is not perfect with probability $p \in [0, 1)$. The conditional ROCOF of the system remains undisturbed under minimal repair and imperfect PM. Beichelt & Fischer (1980) consider two types (I & II) of failures and present a generalized age replacement policy in which type I failures are minimally repaired, and the system is replaced when a type II failure is observed or when the system age reaches T, whichever is earlier. Boland & Proschan (1982) relax the constant minimal repair cost assumption of Barlow & Hunter (1960) and assume that the minimal repair cost increases with the number of minimal repairs since the last PM (or replacement). Optimal PM policies are then presented for both finite and infinite horizon problems when k^{th} minimal repair cost c_k (since last

PM) has the form $c_k = a + kc$, i.e., each additional minimal repair costs c units more than the previous minimal repair.

Nguyen & Murthy (1981) introduce a sequential PM policy with minimal repairs at failures when the conditional ROCOF changes after a PM. The authors assume that PMs are not perfect and the conditional ROCOF of a repairable system increases with number of PMs and time. In other words, if $\lambda_i(t)$ denotes the conditional ROCOF at time t since the last PM of a system subjected to $i-1$ PMs, then

$$\lambda_{i+1}(t) \geq \lambda_i(t), \text{ and } \lambda_i(t) > \lambda_i(s), \ t > s > 0, \ i = 1, 2, 3 \ldots .$$

The system age is reset to 0 after each PM and the system undergoes PMs at ages $T_1, T_2, T_3, \ldots, T_{j-1}$. The system is always replaced after $j-1$ PMs at age T_j. The optimal maintenance policy is derived over decision variables j and $(T_1, T_2, \ldots, T_j)$ that minimizes the expected maintenance cost per unit time of a replacement cycle. Nguyen & Murthy (1981) show that a necessary condition for the optimal sequential PM policy is

$$\lambda_i(T_i) = \lambda_1(T_1) \ , \ i = 2, \ldots, j \ ,$$

i.e., the conditional ROCOF at optimal PM ages are the same. Nakagawa (1988, model A) states that it may be unreasonable to assume that conditional ROCOF of a repairable system changes between the PMs and presents a sequential PM model in which,

$$\lambda_{i+1}(t) = a_i \lambda_i(t) \ , \ a_i > 1 \ , \ i = 1, 2, 3 \ldots \ ,$$

where a_i is called the improvement factor in conditional ROCOF after the i^{th} PM.

A necessary condition for the optimal sequential PM policy in this case is

$$A_{i-1}\lambda_1(T_i) = \lambda_1(T_1) \ , \ i = 2, \ldots, j \ ,$$

where

$$A_i = \prod_{k=1}^{i} a_k \ ,$$

which is the same as reported by Nguyen & Murthy (1981), since

$$A_{i-1}\lambda_1(T_i) = a_{i-1}a_{i-2}\cdots a_1\lambda_1(T_i) = \lambda_i(T_i).$$

Nakagawa (1988, model B) also presents another variant of an imperfect sequential PM policy in which the i^{th} PM improves age of the system by a factor $(1-b_i) \in (0,1)$, where b_i is called the *age reduction factor*. In other words, if T_i denotes the age of the system just before the i^{th} PM then the age of the system immediately after the PM reduces to b_iT_i. The optimal PM policy is derived assuming that age reduction factors increase with number of PMs, i.e., $b_{i+1} > b_i$. Chun (1992) analyzes a special case of the age-reduction policy of Nakagawa (1988) by assuming that PMs are performed at equal intervals and each PM reduces the age of the system by a constant. In other words, $T_i = T_1, \forall i$, and therefore, the policy reduces to a constant age improvement periodic PM policy with

$$b_i = \frac{ib_1}{1+(i-1)b_1} \ , \ i = 2, 3, \ldots \ .$$

1.4.2 Maintenance Models for a System of Equipment

In a system, not all the equipment operate independent of each other. There may exist interactions between some of the associated pieces of equipment in the system.

The interactions can be economic, structural or probabilistic. As a result, optimal maintenance decisions obtained from independent analysis of different types of equipment are rarely optimal for the entire system. In general, optimal maintenance policies for such a system are complex and cannot be described easily. Therefore, many researchers attempt to identify problem structures where single-equipment-type maintenance policies are optimal for the entire system, while others look for good maintenance policies that are easy to implement, which may be suboptimal (Thomas 1986).

The research on maintenance models for a system of multiple types of equipment focuses primarily on modeling economic, stochastic and structural interactions between the associated pieces of equipment. Such interactions have been modeled in a variety of different ways in the literature. In the most common setting, economic interactions show up whenever there is a fixed cost of initiating a maintenance activity. In such cases, grouping of maintenance activities is beneficial, resulting in positive economic interactions. On the other hand, negative economic interactions come into play when grouping maintenance results in higher costs e.g., due to higher production losses, safety concerns or manpower constraints. Stochastic interactions (also known as failure interactions) occur when failure of one unit of equipment affects the lifetime distribution of other pieces of equipment in the system. For example, failure of one piece of equipment may cause another to operate under higher loads, altering its failure characteristics. Structural interactions are common when PM of one unit of equipment requires dismantling one or more pieces of working equipment, thus creating an opportunity to perform PM on the set of dismantled equipment.

We next review relevant work on optimal maintenance models for a system of multiple types of equipment. The reader is referred to excellent survey articles by Thomas (1986), Cho & Parlar (1991), Dekker et al. (1997), Wang (2002), and Nicolai & Dekker (2006) for additional work in this area.

Okumoto & Elsayed (1983) investigate a group maintenance policy that minimizes the sum of production losses and repair costs for a system of N independent but identical machines, which is repaired when the system attains an age of T units. The repair cost has fixed and variable cost components. The authors assume that repair actions bring the system to as good as new condition but do not consider the cost of PM for the set of operational machines at the time of repair. Assaf & Shanthikumar (1987) address a similar problem and assume that repair actions only bring the failed machines to as good as new condition. They provide an optimal policy for the case when machines have identical exponentially distributed lifetimes. The optimal policy initiates the repair when the number of failed machines reach an optimal threshold m.

Ritchken & Wilson (1990) combine the two policies described above and consider an (m, T) group maintenance policy, that advocates repairing the system at the m^{th} failure or at the age T, whichever is earlier. Popova & Wilson (1999) extend m, T and (m, T) group maintenance policies for the case when lifetime of a machine has a phase type distribution. Haurie & L'Ecuyer (1982) allow the replacement of a subset of working machines in addition to the set of failed machines, at the time system repair is initiated. They present a dynamic programming approach to find the optimal maintenance policies, which are counterintuitive in some cases. For example, let $n_{\mathbf{t}}$ denote the optimal number of machines replaced when the state of the system is given by age vector $\mathbf{t}$, then the optimal replacement policy does not always imply $n_{\mathbf{t_1}} \geq n_{\mathbf{t_2}}$ if $\mathbf{t_1} \geq \mathbf{t_2}$.

Murthy & Nguyen (1985b) are the first to consider stochastic interactions between associated components of a system. They introduce three different types of failure interactions for a system with two components. In a type I interaction a failure of the first component can trigger failure of the second component with probability p. Similarly, a failure of the second component can induce a failure of the first component with probability q. A type II interaction allows failure of the first component to induce a failure of the second component with some probability but the failure of the second component acts as a shock to the first component. Therefore, in a type II interaction, failure of the second component affects the conditional ROCOF of the first component without inducing an instantaneous failure. In a type III failure interaction, every failure of the first component acts as a shock to the second component and vice versa. In other words, for a type III interaction there are no instantaneous induced failures.

Murthy & Nguyen (1985b) investigate a system of two components with type I failure interaction, and report expected total cost and expected cost per unit time of the system when it is allowed to operate for finite and infinite time, respectively. Murthy & Nguyen (1985a) extend their analysis to a system of multiple types of equipment with type I failure interaction and investigate two maintenance policies. The first policy calls for replacement of the single failed component when there is no system failure, whereas under the second policy, failed component undergoes minimal repair. In both policies the system is always replaced whenever there is a total system failure. Lai & Chen (2006) consider a two-unit system where the failure of each unit either increases the conditional ROCOF of the other or brings it to an instantaneous failure. The system is replaced at age T or at failure, whichever occurs first. The value of T is obtained by minimizing the long-run expected cost per unit time.

In general, maintenance problems in the presence of such interactions are difficult to model and solve. However, if there are no such interactions, maintenance policies for the entire system can be derived from independent analysis of each component in the system.

Chapter 2

Bayesian Approach to Parameter Estimation in Maintenance Planning

In this chapter, we focus on estimating two key sets of parameters that play a vital role in maintenance planning. One set describes the effectiveness of maintenance and the other defines the conditional ROCOF. In a NPP setting, the equipment data are heavily right censored and in some cases there are not enough failure data. In such situations, the notion of data speaking for itself does not hold. As a result, classical estimation procedures such as maximum likelihood (ML), methods of moments etc., do not work well against the idiosyncrasies of the small datasets. Bayesian methods are known to work well for such situations, where the prior knowledge of system operators can be combined with the equipment failure data to estimate the parameters of interest.

In this research, we focus on a parametric form of the conditional ROCOF function. We assume that the parameters of the conditional ROCOF and parameters describing the quality of maintenance actions are random variables with a prior distribution. We use Bayes' theorem to compute the posterior distribution for the parameters in light of observed data that is used to compute the likelihood function.

2.1 Imperfect Maintenance

In the reliability literature, CMs are typically modeled as *minimal repairs*. Minimal repair by definition restores the equipment to its condition immediately before the failure. In other words, minimal repair brings the equipment to an *as good as old* condition. This modeling assumption may be perfectly valid for equipment with many parts in which, failure of any part may result in equipment failure. Therefore, operational failures of such types of equipment are generally fixed by replacing or repairing a few parts which do not affect the failure behavior of the equipment as a whole. In contrast, it is common to assume that a PM makes the equipment *as good as new*, which once again is a valid assumption if the PM results in equipment replacement or a complete overhaul (see Barlow & Hunter 1960). In what follows, we use the term *idealized view* to refer to the assumption of perfect PMs and minimal repair CMs.

Over the past five decades, various optimal maintenance policies have been reported based on the *idealized view* of CM and PM. In practice, PM and CM activities overlap to a great extend. For example, at STP it is quite common to perform a CM by following a standard PM procedure. In such situations, the two extreme views on the effectiveness of maintenance may not be suitable for planning PM.

In recent years, researchers have focused on the idea of imperfect maintenance. Imperfect maintenance has been modeled in a variety of different ways (see Pham & Wang 1996, for more details). In the most general setting, the state of the equipment after an imperfect maintenance is no better than new equipment. Therefore, it is possible that the equipment after an imperfect maintenance is in a state, worse than the state prior to the maintenance. Pham & Wang (1996) refer to such maintenance actions as *worse* and *worst* repairs, where in the latter case equipment fails immediately after the maintenance. We would like to clarify that both *worse* and *worst* repairs are not intentional, and are mainly due to human error and incorrect maintenance procedures.

2.1.1 Concept of Virtual Age

In this research, we use the notion of *virtual age* to model imperfect maintenance. Kijima et al. (1988) first introduce the idea of *virtual age*, a term generally used to differentiate from the equipment's calendar age. Let $v(t)$ be the virtual age of the equipment at calendar time t. If we assume that both CM and PM are instantaneous, then the equipment is always in a working state. Therefore, at a given time t the equipment has survived $v(t)$ units of time, and the probability of a failure given the equipment history $\mathcal{H}_{t^-}$, up to time t, can be written as,

$$\lambda_c(t \mid \mathcal{H}_{t^-}) = r(v(t)), \tag{2.1}$$

where, $r(\cdot)$ is the failure rate of the distribution of the time to first equipment failure, say X (Rausand & Høyland 2004). Equation (2.1) establishes the relationship between the conditional failure rate (conditional ROCOF) of the *counting process* $\{\mathcal{N}(t), t \geq 0\}$ and the failure rate function $r(\cdot)$ of X, and is key to the analysis of repairable equipment.

2.1.2 Age Reduction Factors

We model the effectiveness of maintenance actions through *age reduction factors*. Kijima et al. (1988) introduce the idea of reduction in virtual age through age reduction factors. Under this concept, a given maintenance action removes a portion of the accumulated age since the last repair but it does not affect the virtual age of the equipment prior to the last repair. We refer to this age-reduction model as *Kijima-I* and it can be formalized as follows.

Consider a piece of equipment that undergoes maintenance (CM or PM) at time $t = t_1$ and let $v(t_1)$ be its virtual age following the maintenance. The equipment accumulates x units of age before the next maintenance at $t = t_2$, where $x = v(t_2^-) - v(t_1)$, and $v(t_2^-)$ and $v(t_2)$ are the virtual ages of the equipment immediately before and after the maintenance at $t = t_2$, respectively (see Figure: 2.1). Then in the *Kijima-I* model,

$$v(t_2) = v(t_1) + bx,$$

where, b is the age reduction factor associated with the maintenance action at $t = t_2$. Kijima et al. (1988) assume that $b \in [0, 1]$ and $x = t_2 - t_1$, and present a periodic

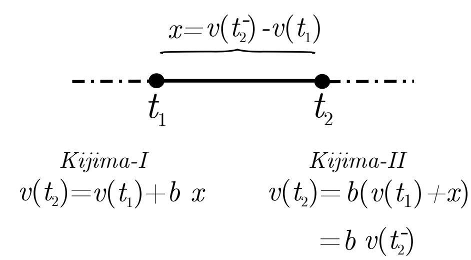

Figure 2.1: Kijima-I and Kijima-II age-reduction models.

replacement policy, where all the intervening repairs have the same age reduction factor. Later, Kijima (1989) introduces a second age-reduction model, we call it *Kijima-II*, in which a given maintenance action removes a portion of the virtual age prior to the maintenance. Therefore, in the *Kijima-II* model the virtual age of the equipment after the maintenance action at t_2 is given by,

$$v(t_2) = b(v(t_1) + x) = bv(t_2^-).$$

It is important to note that Kijima et al. (1988) and Kijima (1989) define age-reduction models only for CMs. Kijima (1989) generalizes the two age-reduction models by allowing the age reduction factor associated with a CM to be an independent random variable over the interval $[0,1]$. The author obtains various bounds associated with the counting process $\{\mathcal{N}(t), t \geq 0\}$ under the periodic *replacement* policy. Nakagawa (1988, model B) on the other hand, associates age reduction factors with PMs without explicitly using the notion of virtual age, and presents an optimal *sequential PM* policy under minimal repair. Jack (1998) and Yu et al. (2008) generalize age-reduction models for both CMs and PMs.

2.2 Likelihood Function

The two key elements in a Bayesian approach to parameter estimation are the likelihood function and prior distribution. In this section, we study the likelihood function for a piece of equipment which is observed over k PM intervals and undergoes CMs whenever it fails during a given PM interval (see Figure 2.2). We assume

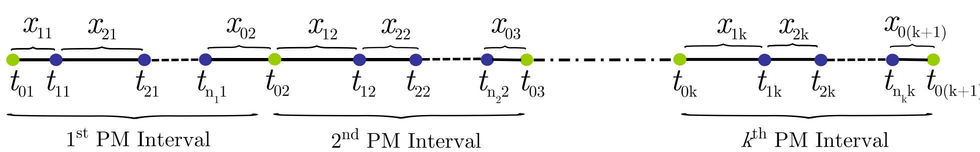

Figure 2.2: Sample maintenance data for k PM intervals. The lime and the blue dots represent PM and CM times, respectively.

that the time to perform PMs and CMs is negligible compared to the length of the period over which the equipment behavior is observed, and it can be ignored for mathematical convenience. We consider that both PM and CM actions are imperfect and let θ_{PM} and θ_{CM} be the associated age reduction factors, respectively. In addition, we use the following notation adopted from Jack (1998):

$t_{0(j+1)}$: time of j^{th} PM; $j = 1, 2, \ldots, k$

t_{ij} : time of i^{th} failure (CM) in the j^{th} PM interval; $i = 1, 2, \ldots, n_j$

$x_{0(j+1)} = t_{0(j+1)} - t_{n_j j}$

time between j^{th} PM and ${n_j}^{\text{th}}$ maintenance in the j^{th} PM interval

$x_{ij} = t_{ij} - t_{(i-1)j}$

time between i^{th} CM and $(i-1)^{\text{st}}$maintenance in the j^{th} PM interval

$v_{0(j+1)}$: virtual (effective) age following j^{th} PM

v_{ij} : virtual (effective) age following i^{th} CM in the j^{th} PM interval

Let $X_{0(j+1)}$ and X_{ij} be the random variables associated with $x_{0(j+1)}$ and x_{ij}, respectively. We note that random variable X_{11} represents the time to the first equipment failure and let $r_{X_{11}}(\cdot)$ be the associated failure rate function. In what follows, we simply use $r(\cdot)$ to represent $r_{X_{11}}(\cdot)$, for notational convenience. We consider a parametric form of the failure rate function $r(\cdot)$ and denote its parameters by a vector $(\theta_1, \theta_2, \ldots, \theta_q)$. For general values of θ_{PM} and θ_{CM}, the random variables X_{ij} are neither independent nor identically distributed. Therefore, well established statistical tools for analysis of i.i.d. observations cannot be used to estimate $(\theta_1, \theta_2, \ldots, \theta_q)$. For a similar reason, non-parametric estimates of $r(\cdot)$ such as those due to Nelson–Aalen and Kaplan–Meier (see Rausand & Høyland 2004) which assume i.i.d. random variables, cannot be used.

Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q, \theta_{\text{PM}}, \theta_{\text{CM}})$. We denote the virtual age of the equipment at time t by $v(t)$, which is given by the following relationship:

$$v(t) = \begin{cases} v_{(i-1)j} + t - t_{(i-1)j} & \text{if } \ t_{(i-1)j} \le t < t_{ij} \\ v_{n_j j} + t - t_{n_j j} & \text{if } \ t_{n_j j} \le t < t_{0(j+1)}. \end{cases} \tag{2.2}$$

Under the *Kijima-I* age-reduction model for both CM and PM, we have

$$v_{ij} = v_{(i-1)j} + \theta_{\text{CM}}(t_{ij} - t_{(i-1)j}), \quad \text{and} \tag{2.3}$$

$$v_{0(j+1)} = v_{n_j j} + \theta_{\text{PM}}(t_{0(j+1)} - t_{n_j j}). \tag{2.4}$$

Similarly, under the *Kijima-II* age-reduction model,

$$v_{ij} = \theta_{\text{CM}}(v_{(i-1)j} + t_{ij} - t_{(i-1)j}), \text{ and} \tag{2.5}$$

$$v_{0(j+1)} = \theta_{\text{PM}}(v_{n_j j} + t_{0(j+1)} - t_{n_j j}). \tag{2.6}$$

When $\theta_{\text{CM}} = 1$, the counting process $\{\mathcal{N}(t), t \geq 0\}$ within each PM interval is a non-homogeneous Poisson process (NHPP) under both *Kijima-I* and *Kijima-II* virtual age-reduction models. Furthermore, $\theta_{\text{PM}} = 0$ in *Kijima-I* does not necessarily mean that the PMs result in *as good as new* equipment. In this case, the virtual age following a PM is simply the virtual age following the last PM or CM. Jack (1998) modifies the definition of the virtual age following a PM under *Kijima-I* and uses

$$v_{0(j+1)} = v_{0j} + \theta_{\text{PM}}(v_{n_j j} - v_{0j} + t_{0(j+1)} - t_{n_j j}), \tag{2.7}$$

instead of equation (2.4). We call this the *modified Kijima-I* age-reduction model and note that under this model, the counting process $\{\mathcal{N}(t), t \geq 0\}$ is once again a NHPP (within each PM interval), when $\theta_{\text{CM}} = 1$. In addition, $\theta_{\text{PM}} = 0$ ensures that PMs result in *as good as new* equipment.

Proposition 2.2.1 (Anderson et al. 1993, section II.7) *The likelihood function of $\boldsymbol{\theta}$ for $\mathbf{t} = \big(t_{11}, t_{21}, \ldots\ldots, t_{n_k k}, t_{0(k+1)}\big)$ is given by:*

$$\mathcal{L}\left(\boldsymbol{\theta} \mid \mathbf{t}\right) = \exp\left(-\int_0^{t_{0(k+1)}} r(v(t))\mathrm{d}t\right) \prod_{j=1}^{k} \prod_{i=1}^{n_j} r\big(v_{(i-1)j} + t_{ij} - t_{(i-1)j}\big)\,. \tag{2.8}$$

Proof: To prove Proposition 2.2.1, we first show that the likelihood function of $\boldsymbol{\theta}$ in terms of $\mathbf{x} = \big(x_{11}, x_{21}, \ldots\ldots, x_{n_k k}, x_{0(k+1)}\big)$ is:

$$\mathcal{L}\left(\boldsymbol{\theta} \mid \mathbf{x}\right) = \prod_{j=1}^{k} \left[\exp\left(-\int_{v_{n_j j}}^{v_{n_j j}+x_{0(j+1)}} r(u)\mathrm{d}u\right) \prod_{i=1}^{n_j} r\big(v_{(i-1)j} + x_{ij}\big) \exp\left(-\int_{v_{(i-1)j}}^{v_{(i-1)j}+x_{ij}} r(u)\mathrm{d}u\right)\right]. \tag{2.9}$$

Let $f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta})$ denote the joint distribution of the random sample $\mathbf{X} = (X_{11}, X_{21}, \ldots, X_{n_k k}, X_{0(k+1)}\big)$ given the parameter $\boldsymbol{\theta}$. Then,

$$\begin{aligned}
\mathcal{L}\left(\boldsymbol{\theta} \mid \mathbf{x}\right) &= f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}) \\
&= \prod_{j=1}^{k} \Bigg[\left(1 - F_{X_{0(j+1)} \mid X_{11}, X_{21}, \ldots\ldots, X_{n_j j}, \boldsymbol{\Theta}}\left(x_{0(j+1)} \mid x_{11}, x_{21}, \ldots\ldots, x_{n_j j}, \boldsymbol{\theta}\right)\right) \\
&\qquad \prod_{i=1}^{n_j} f_{X_{ij} \mid X_{11}, X_{21}, \ldots\ldots, X_{(i-1)j}, \boldsymbol{\Theta}}\left(x_{ij} \mid x_{11}, x_{21}, \ldots\ldots, x_{(i-1)j}, \boldsymbol{\theta}\right)\Bigg]\,.
\end{aligned} \tag{2.10}$$

In the last step, we write conditional distributions using the multiplicative rule and since the $x_{0(k+1)}$ represent censored observations, we use the survival function instead of the distribution function for these observations.

Now for all $j = 1, 2, \ldots, k$, and $i = 1, 2, \ldots, n_j$, we have

$$\begin{aligned}
&f_{X_{ij} \mid X_{11}, X_{21}, \ldots\ldots, X_{(i-1)j}, \boldsymbol{\Theta}}\left(x_{ij} \mid x_{11}, x_{21}, \ldots\ldots, x_{(i-1)j}, \boldsymbol{\theta}\right) \\
&\quad = \lim_{\Delta x \to 0} \frac{\mathbb{P}\left\{x_{ij} \le X_{ij} < x_{ij} + \Delta x \mid x_{11}, x_{21}, \ldots\ldots, x_{(i-1)j}, \boldsymbol{\theta}\right\}}{\Delta x}.
\end{aligned}$$

Since $\left(x_{11}, x_{21}, \ldots\ldots, x_{(i-1)j}, \boldsymbol{\theta}\right)$ represents new equipment that has survived $v_{(i-1)j}$ time units, we have

$$
\begin{aligned}
& f_{X_{ij} \mid X_{11}, X_{21}, \ldots\ldots, X_{(i-1)j}, \boldsymbol{\Theta}}\left(x_{ij} \mid x_{11}, x_{21}, \ldots\ldots, x_{(i-1)j}, \boldsymbol{\theta}\right) \\
& \quad = \lim_{\Delta x \to 0} \frac{\mathbb{P}\left\{v_{(i-1)j} + x_{ij} \leq X_{11} < v_{(i-1)j} + x_{ij} + \Delta x \mid X_{11} > v_{(i-1)j}\right\}}{\Delta x} \\
& \quad = \lim_{\Delta x \to 0} \frac{\mathbb{P}\left\{v_{(i-1)j} + x_{ij} \leq X_{11} < v_{(i-1)j} + x_{ij} + \Delta x\right\}}{\Delta x \qquad \mathbb{P}\left\{X_{11} > v_{(i-1)j}\right\}} \\
& \quad = \frac{f_{X_{11}}\left(v_{(i-1)j} + x_{ij}\right)}{1 - F_{X_{11}}\left(v_{(i-1)j}\right)} \\
& \quad = \frac{r\left(v_{(i-1)j} + x_{ij}\right) \quad \exp\left(-\int_0^{v_{(i-1)j} + x_{ij}} r(u) \mathrm{d}u\right)}{\exp\left(-\int_0^{v_{(i-1)j}} r(u) \mathrm{d}u\right)} \\
& \quad = r\left(v_{(i-1)j} + x_{ij}\right) \quad \exp\left(-\int_{v_{(i-1)j}}^{v_{(i-1)j} + x_{ij}} r(u) \mathrm{d}u\right). \qquad (2.11)
\end{aligned}
$$

Similarly, for $j = 1, 2, \ldots, k$,

$$
\begin{aligned}
& 1 - F_{X_{0(j+1)} \mid X_{11}, X_{21}, \ldots\ldots, X_{n_j j}, \boldsymbol{\Theta}}\left(x_{0(j+1)} \mid x_{11}, x_{21}, \ldots\ldots, x_{n_j j}, \boldsymbol{\theta}\right) \\
& \quad = \mathbb{P}\left\{X_{11} > v_{n_j j} + x_{0(j+1)} \mid X_{11} > v_{n_j j}\right\} \\
& \quad = \exp\left(-\int_{v_{n_j j}}^{v_{n_j j} + x_{0(j+1)}} r(u) \mathrm{d}u\right). \qquad (2.12)
\end{aligned}
$$

Substituting (2.11) and (2.12) in (2.10), we obtain (2.9). To prove this proposition, we substitute u with $v(t)$, where $v(t)$ is defined in equation (2.2). We can then re-write the likelihood function in (2.9) as:

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\theta} \mid \mathbf{t}) &= \prod_{j=1}^{k}\left[\exp\left(-\int_{t_{n_j j}}^{t_{0(j+1)}} r(v(t)) \mathrm{d}t\right) \prod_{i=1}^{n_j} r\left(v_{(i-1)j} + t_{ij} - t_{(i-1)j}\right) \exp\left(-\int_{t_{(i-1)j}}^{t_{ij}} r(v(t)) \mathrm{d}t\right)\right] \\
&= \exp\left(-\int_0^{t_{0(k+1)}} r(v(t)) \mathrm{d}t\right) \prod_{j=1}^{k} \prod_{i=1}^{n_j} r\left(v_{(i-1)j} + t_{ij} - t_{(i-1)j}\right). \qquad \square
\end{aligned}
$$

Jack (1998) uses the likelihood function expressed in (2.8) in his work. In addition, Yu et al. (2008) construct a similar expression for the likelihood function provided in equation (2.9). Finally, we note that for identical equipment, the corresponding likelihood is just a product of individual likelihoods.

2.2.1 Maximum Likelihood Estimation

Let $\boldsymbol{\theta}^*$ denote the value of $\boldsymbol{\theta}$ for which $\mathcal{L}(\boldsymbol{\theta} \mid \mathbf{x})$ attains its maximum for a fixed $\mathbf{x}$. Mathematically, we write

$$\mathcal{L}(\boldsymbol{\theta}^* \mid \mathbf{x}) = \max_{\boldsymbol{\theta}} \ \mathcal{L}(\boldsymbol{\theta} \mid \mathbf{x}),$$

or equivalently,

$$\mathcal{L}(\boldsymbol{\theta}^* \mid \mathbf{t}) = \max_{\boldsymbol{\theta}} \ \mathcal{L}(\boldsymbol{\theta} \mid \mathbf{t}).$$

Then $\boldsymbol{\theta}^*$ is known as a *maximum likelihood estimate* (MLE) of $\boldsymbol{\theta}$ and is simply a point in the parameter space which maximizes the probability of the observed sample. In general, MLE converge almost surely to the true value of the parameter but are known to be numerically sensitive for small datasets (see Casella & Berger 2002).

2.3 Choice of Priors

In Bayesian analysis, the *likelihood* of observing a sample $\mathbf{x}$ for a given realization of parameters is combined with the *prior* distribution on the parameters and summarized in a *posterior* distribution. Let denote $\pi_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ the prior distribution on Θ and $\pi_{\boldsymbol{\Theta} \mid \mathbf{X}}(\boldsymbol{\theta} \mid \mathbf{x})$ be the posterior distribution of interest. Then according to Bayes' rule,

$$\pi_{\Theta|\mathbf{X}}(\boldsymbol{\theta} \mid \mathbf{x}) \propto \pi_{\Theta}(\boldsymbol{\theta})\mathcal{L}(\boldsymbol{\theta} \mid \mathbf{x}).$$

Prior distributions reflect a decision maker's belief in the parameters ($\boldsymbol{\theta}$) before the data ($\mathbf{x}$) are observed. One can use the maximum entropy principle to select a suitable prior but in this research, we rely on the subjective (expert) opinion of system engineers and risk managers at STP. We use a uniform prior for the failure rate

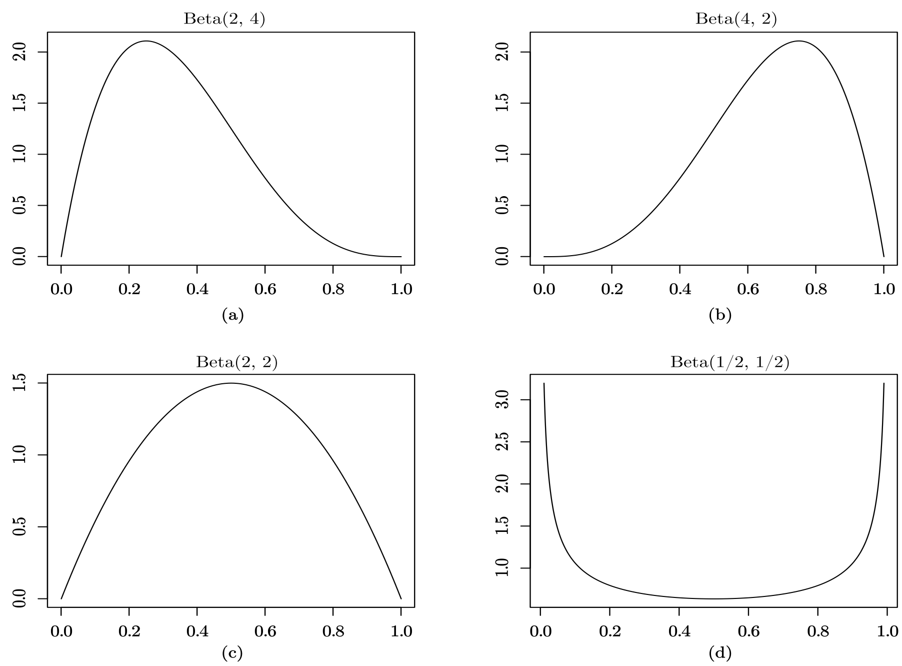

Figure 2.3: Beta priors for age reduction factors.

parameters, and since $\theta_{\text{PM}} \in [0,1]$ and $\theta_{\text{CM}} \in [0,1]$, the beta distribution becomes a natural choice for the age reduction factors. We assume that the parameters of the prior distributions (also called hyper-parameters in the Bayesian setting), are known to domain experts. Figure 2.3 shows different prior distributions that are used for the two age reduction factors and can be interpreted as follows:

(a) An effective maintenance;

(b) A minimal impact maintenance;

(c) A maintenance action of intermediate effectiveness; and,

(d) A maintenance action with divided expert opinion.

2.4 Gibbs Sampler

In this section, we illustrate the use of a Gibbs sampler in summarizing the posterior distribution $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\boldsymbol{\theta} \mid \mathbf{x})$. We first note that due to the complexity of the likelihood function, $\mathcal{L}(\boldsymbol{\theta} \mid \mathbf{x})$ (see equation (2.9)), in general it is difficult to derive an analytical expression for the posterior distribution of the parameters. Therefore, we look for a sampling algorithm that approximates the posterior distribution without directly simulating from it. General purpose algorithms, such as *rejection* and *importance* sampling require suitable *proposal* densities defined on the parameter space, which are difficult to construct for high-dimensional problems. On the other hand, the choice of *proposal* densities in Markov Chain Monte Carlo (MCMC) methods such as a Gibbs sampler and Metropolis-Hastings algorithms are fairly straightforward. Therefore, these methods are well-adapted in Bayesian analysis (see Robert & Casella 2005, for more detail).

Recall that $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q, \theta_{\text{PM}}, \theta_{\text{CM}})$. Suppose we define the following full conditional distributions:

$$
\begin{gathered}
\pi\left(\theta_1 \mid \theta_2, \theta_3, \ldots, \theta_q, \theta_{\text{PM}}, \theta_{\text{CM}}, \mathbf{x}\right) \\
\pi\left(\theta_2 \mid \theta_1, \theta_3, \ldots, \theta_q, \theta_{\text{PM}}, \theta_{\text{CM}}, \mathbf{x}\right) \\
\vdots \\
\pi\left(\theta_q \mid \theta_1, \theta_2, \ldots, \theta_{q-1}, \theta_{\text{PM}}, \theta_{\text{CM}}, \mathbf{x}\right)
\end{gathered}
$$

$$\pi\left(\theta_{\mathrm{PM}} \mid \theta_1, \theta_2, \ldots, \theta_q, \theta_{\mathrm{CM}}, \mathbf{x}\right)$$
$$\pi\left(\theta_{\mathrm{CM}} \mid \theta_1, \theta_2, \ldots, \theta_q, \theta_{\mathrm{PM}}, \mathbf{x}\right).$$

A standard Gibbs sampler then constructs an ergodic Markov chain by sampling successively from the set of full conditional distributions defined above. The limiting distribution of the Markov chain is the joint posterior distribution of the parameters. Unfortunately, in our case it is not convenient to sample from the set of full conditionals due to the complex form of the likelihood function. Therefore, we use a random walk-based Metropolis-Hastings algorithm within Gibbs sampling procedure. For notational convenience, let $\theta_{q+1}= \theta_{\mathrm{PM}}$ and $\theta_{q+2}= \theta_{\mathrm{CM}}$, so that, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q, \theta_{q+1}, \theta_{q+2})$. Let θ_k^n, $k \in \{1, 2, \ldots, q+2\}$ be the value of θ_k at the n^{th} cycle of the Gibbs sampler. Then the candidate value of θ_k is given by

$$\hat{\theta}_k = \theta_k^n + a_k Z,$$

where a_k is a fixed scale factor and Z denote a standard normal random variable. The next simulated value of θ_k, θ_k^{n+1}, is equal to the candidate value, $\hat{\theta}_k$ with probability,

$$P = \min\left\{1, \frac{\pi\left(\hat{\theta}_k \mid \theta_1^{n+1}, \theta_2^{n+1}, \ldots, \theta_{k-1}^{n+1}, \theta_{k+1}^n, \ldots, \theta_{q+2}^n, \mathbf{x}\right)}{\pi\left(\theta_k^n \mid \theta_1^{n+1}, \theta_2^{n+1}, \ldots, \theta_{k-1}^{n+1}, \theta_{k+1}^n, \ldots, \theta_{q+2}^n, \mathbf{x}\right)}\right\}; \tag{2.13}$$

otherwise, $\theta_k^{n+1} = \theta_k^n$ (see Albert 2009). We note that the Gibbs sampler requires an initial value of all the parameters and a vector of scale factors. Since the limiting distribution of an ergodic Markov chain does not depend on the initial state, in theory, we can select any initial value from the parameter space. But in practice, a randomly selected initial value with a poor choice of scale factor, can result in a slower rate of convergence of the Gibbs sampler. On the other hand, it is important to check that MCMC methods converge to the same limiting distribution when the Markov chain is initialized at different values. One can use the method of moments or MLEs to initialize the Gibbs sampler. Also, we can use the inverse of the Hessian matrix of the likelihood function evaluated at the MLE to obtain an estimate of the variance-covariance matrix of the parameters, which can then be used to determine the scale factors, a_k.

2.5 Computational Results

We implement the Gibbs sampler described in the previous section using the statistical package R and provide estimates of the posterior distributions for three datasets. The first two datasets are obtained from the literature and the third is from STP. We assume that the time to the first equipment failure, X_{11}, has a Weibull distribution, which is widely used in reliability analysis. Both of our datasets from the literature assume a Weibull distribution for X_{11}. Recall that if $X_{11} \sim \text{Weibull}\,(\theta_1, \theta_2)$, then for $x > 0$,

$$
\begin{aligned}
F_{X_{11}}(x) &= 1 - \exp\left(-\left(\frac{x}{\theta_2}\right)^{\theta_1}\right), \\
f_{X_{11}}(x) &= \frac{\theta_1}{\theta_2}\left(\frac{x}{\theta_2}\right)^{\theta_1 - 1} \exp\left(-\left(\frac{x}{\theta_2}\right)^{\theta_1}\right),
\end{aligned}
$$

and

$$
r(x) = \frac{\theta_1}{\theta_2}\left(\frac{x}{\theta_2}\right)^{\theta_1 - 1},
$$

where, $\theta_1 > 0$ is called the *shape* and $\theta_2 > 0$ is referred to as the *scale* parameter.

2.5.1 Example 1: Syringe-Driver Infusion Pump Dataset from Baker (1991)

Baker (1991) studies the effectiveness of PMs for a set of 43 identical syringe-driver infusion pumps used in a large teaching hospital. Jack (1997) and Jack (1998) consider a subset of 9 of the 43 syringe-driver infusion pumps and report MLEs for the age reduction factors, $(\theta_{\text{PM}}, \theta_{\text{CM}})$, and the failure rate parameters, (θ_1, θ_2). Jack (1997) and Jack (1998) also provide 95% confidence intervals for all the parameters under the *Kijima-II* and the *modified Kijima-I* age-reduction models. We note that the confidence intervals are justified only when large numbers of CMs and PMs are recorded. Since this part of our research is an extension of Jack (1998), we consider the subset of 9 pumps, for comparison purposes, and assume *Kijima-II* age-reduction model. Our analysis can be easily extended for the *Kijima-I* and the *modified Kijima-I* models.

We use a random walk-based Metropolis-Hastings algorithm within the Gibbs sampler to simulate from the posterior. Let $\boldsymbol{\theta}^{\text{lb}} = \left(\theta_1^{\text{lb}}, \theta_2^{\text{lb}}, \theta_{\text{PM}}^{\text{lb}}, \theta_{\text{CM}}^{\text{lb}}\right)$ and $\boldsymbol{\theta}^{\text{ub}} = \left(\theta_1^{\text{ub}}, \theta_2^{\text{ub}}, \theta_{\text{PM}}^{\text{ub}}, \theta_{\text{CM}}^{\text{ub}}\right)$ be lower and upper bounds on the parameters, respectively. We set $\boldsymbol{\theta}^{\text{lb}} = (1, 0.001, 0, 0)$ and $\boldsymbol{\theta}^{\text{ub}} = (5, 5000, 1, 1)$, and assume an independent uniform prior over the support of each parameter. The Weibull *shape* parameter θ_1 indicates the rate at which equipment deteriorates. The failure rate is constant when $\theta_1 = 1$, whereas $\theta_1 > 1$ and $\theta_1 < 1$ indicate increasing and decreasing failure rates, respectively. For mechanical equipment such as pumps, it is reasonable to assume that the failure rate is not decreasing, therefore we set $\theta_1^{\text{lb}} = 1$. Similarly, practical engineering judgments are used to fix θ_1^{ub} and θ_2^{ub} (see Bloch & Geitner 1994, for typical *shape* parameters of different mechanical equipment). The upper bound on θ_2^{ub} can also be fixed to the maximum possible length of a PM interval under consideration. Since $\theta_{\text{PM}} \in [0,1]$ and $\theta_{\text{CM}} \in [0,1]$, the bounds for these two age reduction factors are straightforward.

We initialize the Gibbs algorithm at two different initial points and run 25,000 Gibbs iterations (cycles) using the scale factor, $\mathbf{a} = (a_1, a_2, a_{\text{PM}}, a_{\text{CM}}) = (0.5, 100, 0.25, 0.25)$. An appropriate value for $\mathbf{a}$ is desirable to ensure adequate exploration over the support of the parameters. It also affects the acceptance rate of each parameter within the Gibbs sampler. For random walk-based Gibbs sampling algorithms, Albert (2009) suggests that acceptance rates between 25% and 45% are considered good. Therefore, $\mathbf{a}$ can be fixed with some trial and error to achieve the desired range of acceptance rates.

Figure 2.4 displays marginal posterior density estimates for each of the parameters when the Gibbs sampler is initialized at (a) $\boldsymbol{\theta}^{\mathbf{0}} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta}^{\mathbf{0}} = \boldsymbol{\theta}^{*} = (2.482, 1057.71, 0.789, 1)$, where $\boldsymbol{\theta}^{*}$ is the MLE of $\boldsymbol{\theta}$. Table 2.1 presents a summary of the marginal distributions of each parameter for the two initial values considered. Figure 2.4 and Table 2.1 together justify that there is no significant difference between the estimates of the posterior distributions obtained when the Gibbs sampler is initialized at these two different values.

Figure 2.5 shows trace plots for the two cases (a) and (b). The trace plot for each of the parameters in both cases looks like random noise, justifying good mixing and exploration of the parameter space. In each case, we observe a very small burn-in period for all the parameters. Acceptance rates for the Gibbs sampler in cases (a) and (b) are (33%, 48%, 33%, 11%) and (32%, 47%, 33%, 11%), respectively. Figure 2.6 graphs the lag n autocorrelation between the simulated draws. Since successive draws of a parameter in the Gibbs sampler are dependent, we notice high correlation values for small lags. The correlation values shrink towards zero for all the parameters at higher lags. Figure 2.7 displays running averages of the simulated parameter values. Since trace plots support good mixing of the Markov chain for each parameter, convergence of the running average suggests that the simulated draws provide a reasonably good approximation of the posterior density. Table 2.2 provides a comparison between the 95% confidence intervals, reported in Jack (1998), and (2.5%, 97.5%) posterior quantiles obtained using two different initial values.

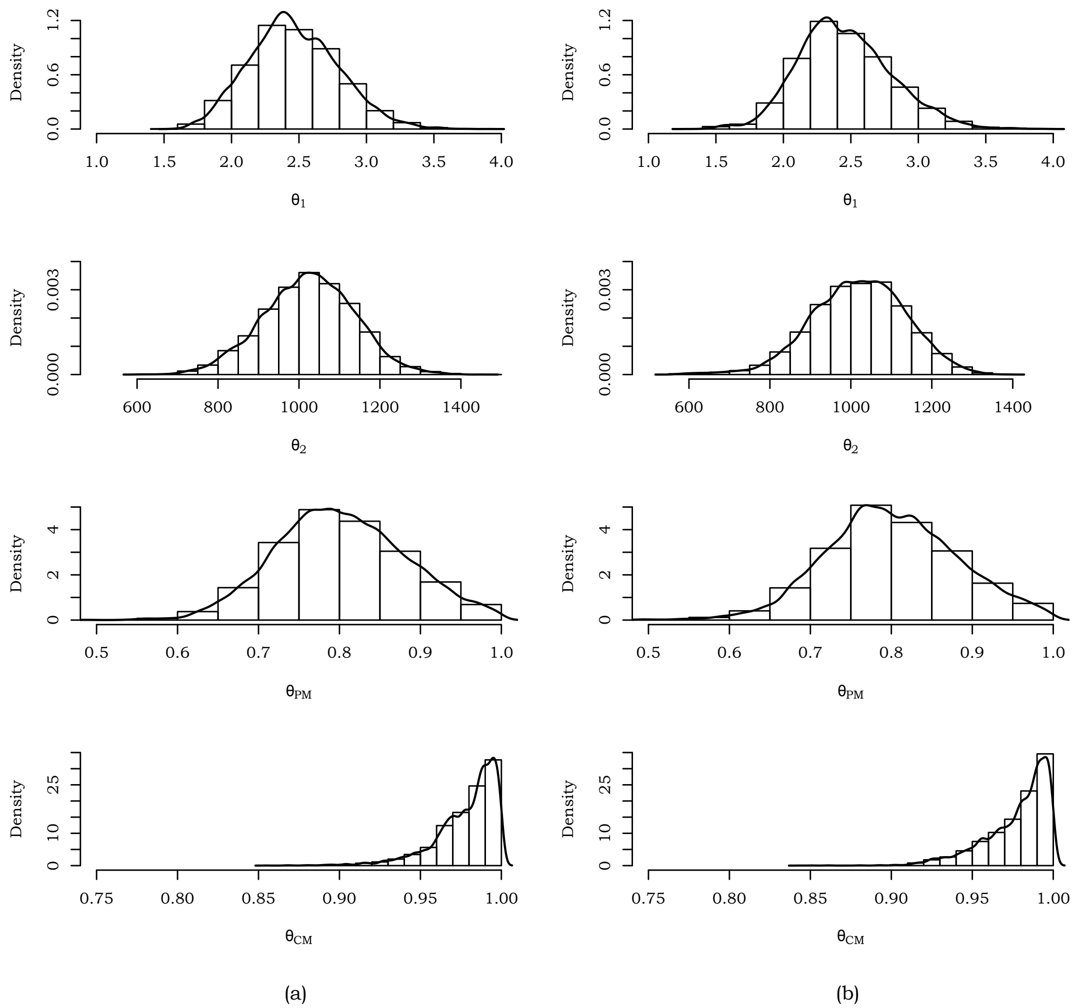

Figure 2.4: Posterior density estimates of parameters when Gibbs sampler is initialized at (a) $\boldsymbol{\theta}^{\mathbf{0}} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta}^{\mathbf{0}} = \boldsymbol{\theta}^{*} = (2.482, 1057.71, 0.789, 1)$.

	(a)							(b)						
	Statistics		Quantiles					Statistics		Quantiles				
	Mean	SD	2.5%	25%	50%	75%	97.5%	Mean	SD	2.5%	25%	50%	75%	97.5%
θ_1	2.465	0.328	1.872	2.235	2.443	2.681	3.139	2.459	0.344	1.854	2.217	2.433	2.674	3.184
θ_2	1024.15	111.92	801.47	949.80	1026.30	1101.11	1239.60	1019.87	115.53	788.28	942.90	1023.23	1099.78	1235.70
θ_{PM}	0.801	0.0792	0.652	0.746	0.798	0.855	0.962	0.801	0.080	0.645	0.749	0.798	0.855	0.962
θ_{CM}	0.979	0.019	0.929	0.969	0.984	0.993	0.999	0.978	0.020	0.928	0.967	0.984	0.993	0.999

Table 2.1: Summary statistics and quantiles of the marginal posterior distribution of parameters for two different initial values (a) $\boldsymbol{\theta^0} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta^0} = \boldsymbol{\theta^*} = (2.482, 1057.71, 0.789, 1)$.

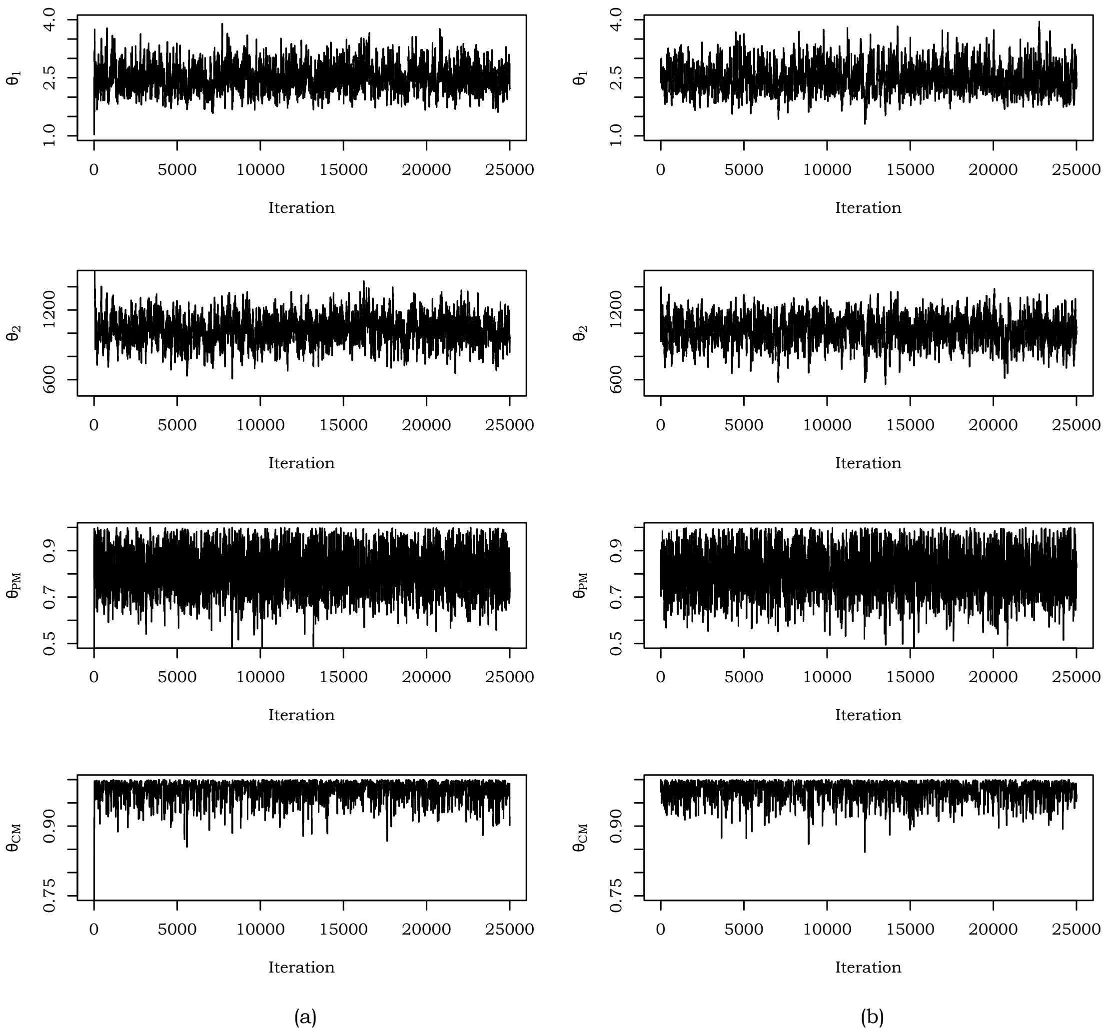

Figure 2.5: Trace plots of simulated parameters in Gibbs sampling displaying good exploration in the parameter space for two different initial values (a) $\boldsymbol{\theta^0} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta^0} = \boldsymbol{\theta^*} = (2.482, 1057.71, 0.789, 1)$.

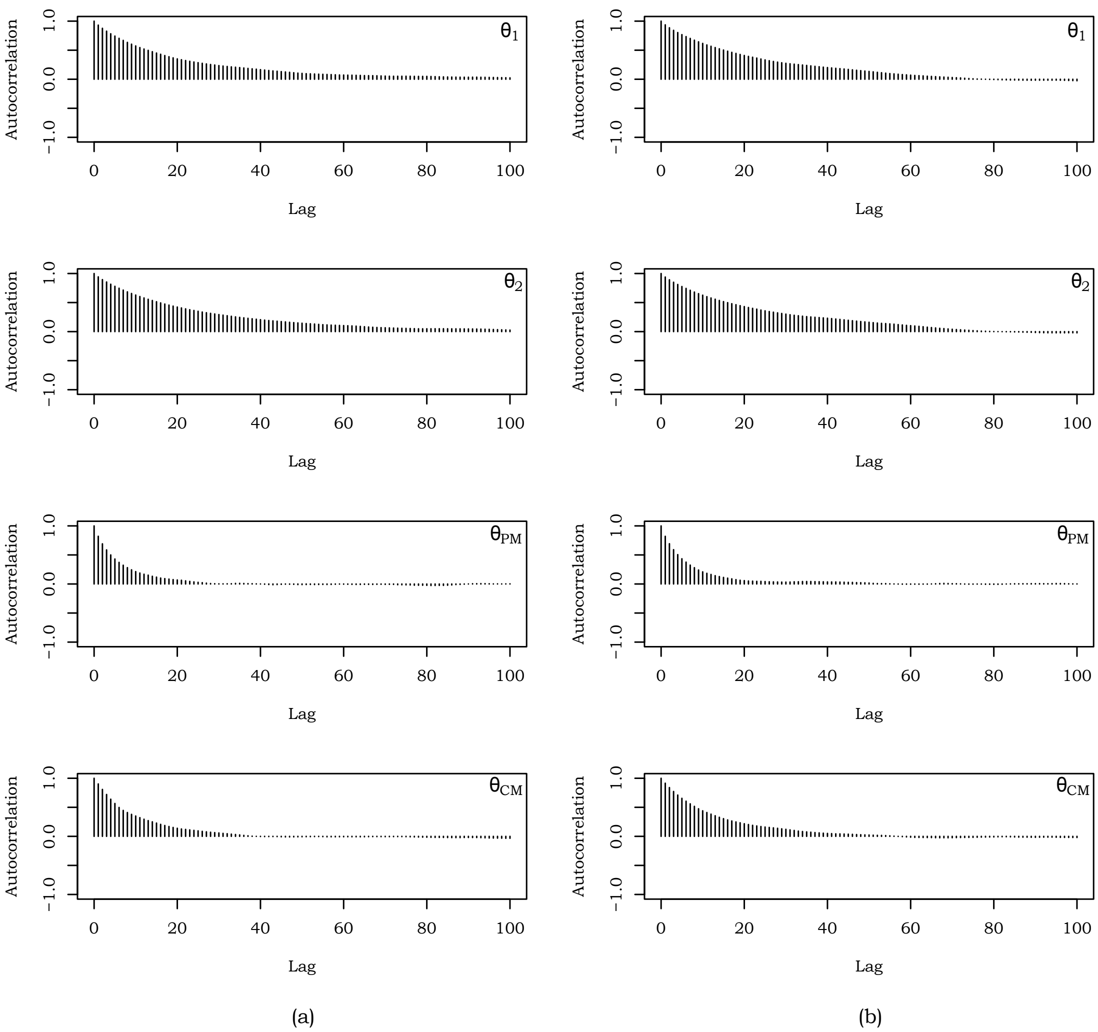

Figure 2.6: Autocorrelation plots of simulated parameters for two different initial values (a) $\boldsymbol{\theta^0} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta^0} = \boldsymbol{\theta^*} = (2.482, 1057.71, 0.789, 1)$.

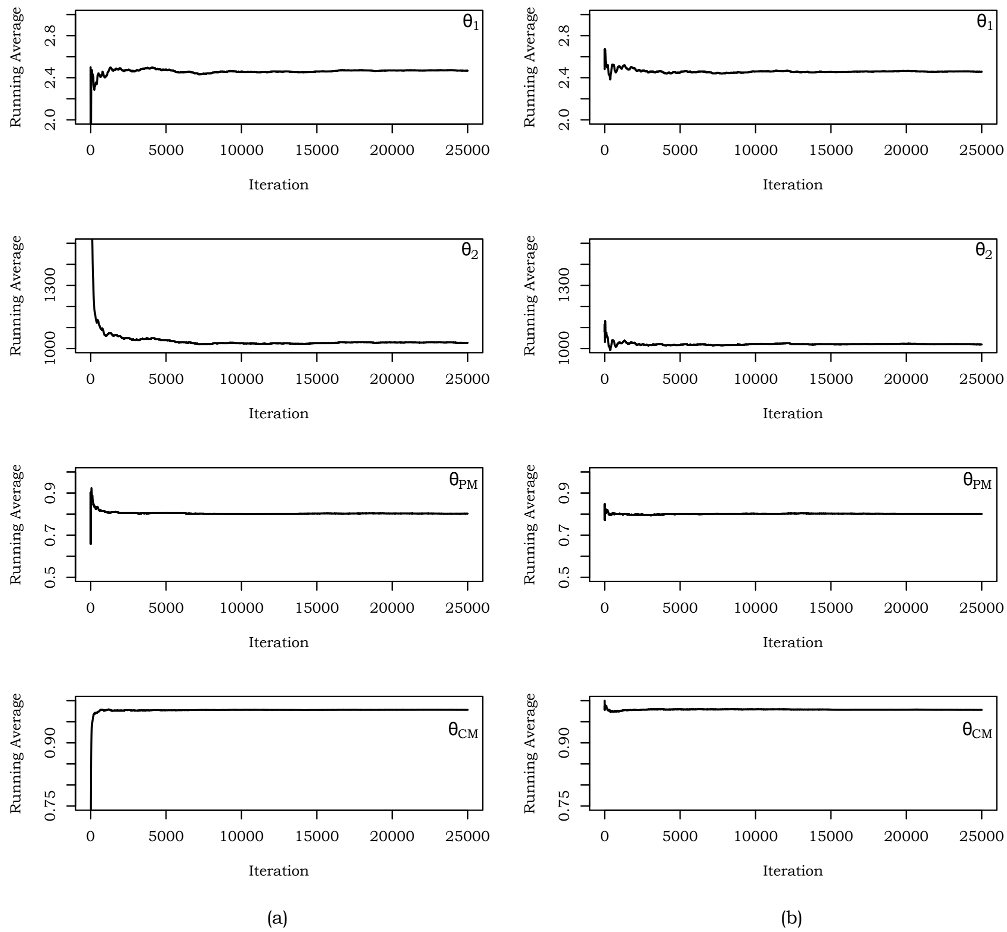

Figure 2.7: Running averages of simulated parameters for two different initial values (a) $\boldsymbol{\theta}^{\mathbf{0}} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta}^{\mathbf{0}} = \boldsymbol{\theta}^{*} = (2.482, 1057.71, 0.789, 1)$.

Parameter	95% Confidence Interval	(2.5%, 97.5%) Posterior Quantiles	
		(a)	(b)
θ_1	(1.87, 3.18)	(1.87, 3.14)	(1.85, 3.18)
θ_2	(826.44, 1272.26)	(801.47, 1239.60)	(788.28, 1235.70)
θ_{PM}	(0.643, 0.987)	(0.652, 0.962)	(0.641, 0.962)
θ_{CM}	(0.896, 1.000)	(0.929, 0.999)	(0.928, 0.999)

Table 2.2: Comparison of confidence intervals reported in Jack (1998) with (2.5%, 97.5%) posterior quantiles obtained using two different initial values (a) $\boldsymbol{\theta}^{\mathbf{0}} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta}^{\mathbf{0}} = \boldsymbol{\theta}^{*} = (2.482, 1057.71, 0.789, 1)$.

2.5.2 Example 2: Simulated Maintenance Dataset from Yu et al. (2008)

Yu et al. (2008) construct a simulated maintenance dataset for a repairable piece of equipment under the *Kijima-I* age-reduction model, and a Weibull distribution for the time to first equipment failure, X_{11}. The dataset is generated using the following set of parameters:

$$\boldsymbol{\theta}^{\text{true}} = \left(\theta_1^{\text{true}}, \theta_2^{\text{true}}, \theta_{\text{PM}}^{\text{true}}, \theta_{\text{CM}}^{\text{true}}\right) = (2.2, 1, 0.8, 0.3)\,.$$

Since $\theta_{\text{CM}}^{\text{true}} < \theta_{\text{PM}}^{\text{true}}$, Yu et al. (2008) implicitly assume that CMs are more effective than PMs. The authors focus on parameter estimation through Bayesian analysis and use a general rejection sampling algorithm, instead of well-adapted variants of Metropolis-Hastings and Gibbs sampling algorithms, to estimate posterior distributions. Their assumption of independent uniform priors on the parameters leads to a simple proposal (envelope) distribution of the form $c\mathcal{L}(\boldsymbol{\theta}^* \mid \mathbf{x})$, uniformly distributed over the parameter space. For $c = 1.1$, an acceptance rate less than 0.001 is reported for the rejection sampling algorithm. We use the dataset provided in their paper and run the Gibbs sampler described in Section 2.4 assuming independent uniform priors on each parameter. The following upper and lower bounds are assumed on the parameters:

$$\begin{aligned} \boldsymbol{\theta}^{\text{lb}} &= (0.001, 0.001, 0, 0)\,, \\ \boldsymbol{\theta}^{\text{ub}} &= (5, 10, 1, 1)\,. \end{aligned}$$

We initialize the Gibbs sampler at $\boldsymbol{\theta}^{\mathbf{0}} = (2.5, 5, 0.5, 0.5)$ and run 25,000 iterations of the algorithm. Figure 2.8 displays the marginal posterior density estimates for each of the parameters. Figure 2.9 suggests good mixing of the Markov chains and Figure 2.10 further suggests that the Gibbs sampler is converging to the true parameters. Table 2.3 presents a comparison of summary statistics and quantiles of the marginal posterior distributions, obtained using Yu et al. (2008)'s rejection sampling algorithm and our Gibbs sampler. We note that the posterior estimates obtained using these two different algorithms are not significantly different. Finally, the acceptance rate of the Gibbs sampler for this dataset is between 30% and 40%, which is much higher than the acceptance rate reported for the rejection sampling algorithm. Hence, the Gibbs sampler is arguably the obvious choice of sampling procedure over the rejection sampling algorithm, for this class of problems.

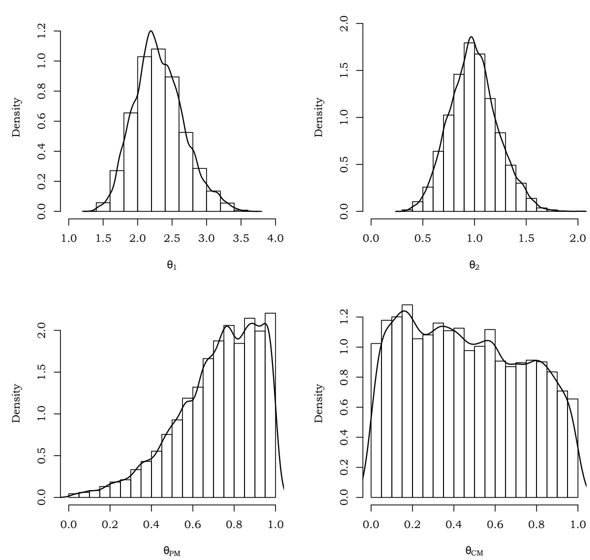

Figure 2.8: Posterior density estimates of the parameters.

2.5.3 Example 3: STP Dataset

In this section, we provide parameter estimates for a real dataset from STP. We consider a dataset from three water transfer pumps, used in the reverse osmosis units

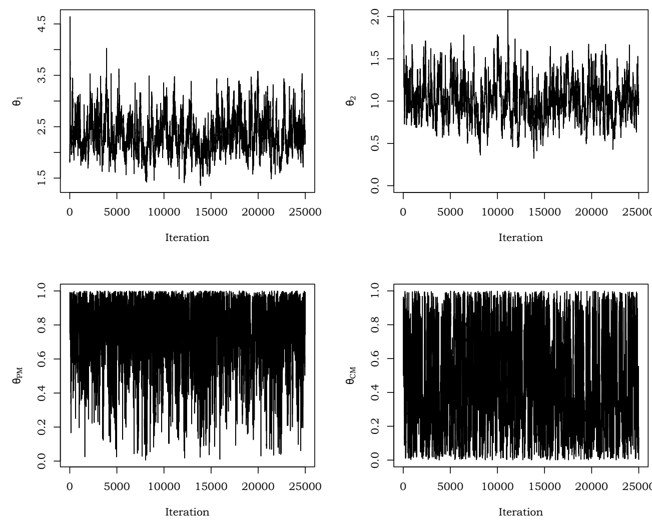

Figure 2.9: Trace plots of simulated parameters in Gibbs sampling

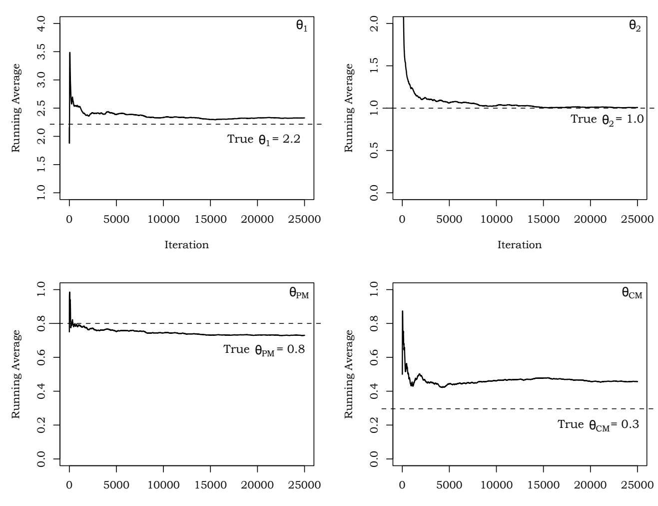

Figure 2.10: Running averages of simulated parameters in Gibbs sampling

	Yu et al. (2008) Results							Gibbs Sampler Results						
	Statistics		Quantiles					Statistics		Quantiles				
	Mean	SD	0%	25%	50%	75%	100%	Mean	SD	0%	25%	50%	75%	100%
θ_1	2.339	-	1.149	1.709	2.311	3.121	4.108	2.311	0.358	1.352	2.065	2.282	2.541	3.626
θ_2	1.031	-	0.237	0.589	1.019	1.547	1.938	0.993	0.232	0.323	0.831	0.984	1.138	2.148
θ_{PM}	0.745	-	0.0009	0.262	0.791	0.992	0.999	0.724	0.197	0.006	0.606	0.757	0.883	1.000
θ_{CM}	0.468	-	0.0001	0.022	0.456	0.969	0.999	0.460	0.280	0.000	0.215	0.439	0.695	0.998

Table 2.3: Comparison of summary statistics and quantiles of the marginal posterior distributions.

of a de-mineralized system. The pumps provide adequate pressure and capacity to the nuclear plant's de-mineralized water distribution system. Currently, only one type of PM is associated with the pumps, which is performed at an interval of 104 weeks (2 years). The PM activities includes draining, flushing and refilling of the bearing housing in an attempt to ensure that the pump's bearing does not fail due to oil contamination and degradation.

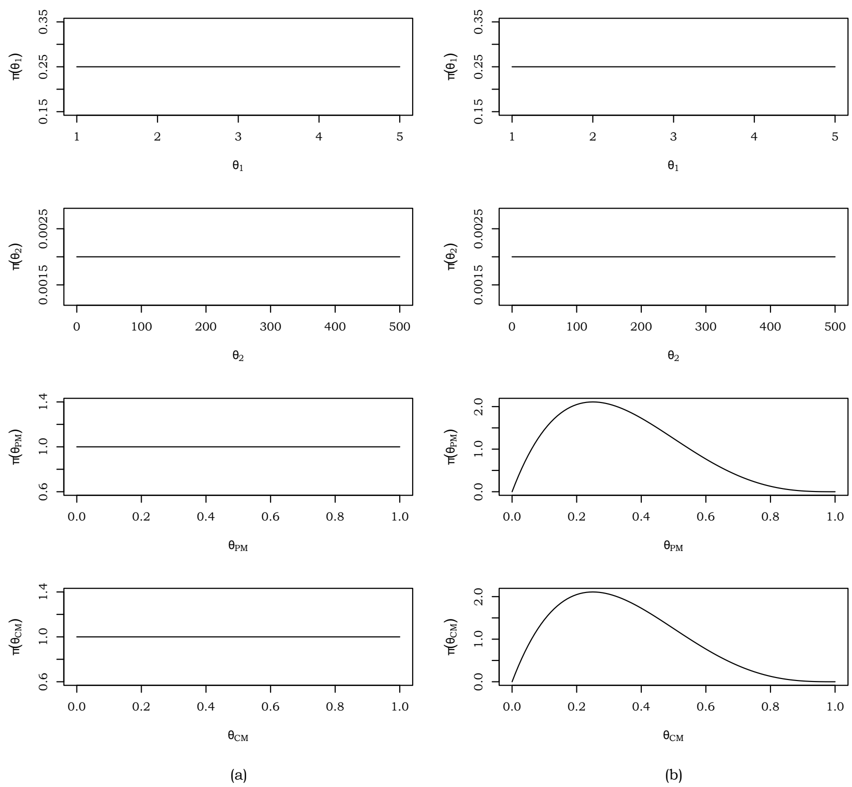

Figure 2.11: Priors densities for all the parameters:(a) Uniform prior with bounded support and (b) uniform priors on the failure rate, $r(\cdot)$, parameters and skew priors on age reduction factors, reflecting the system engineers' belief that both PM and CM actions are effective.

System engineers for the de-mineralized system strongly believe that both PMs and CMs on the water transfer pumps are highly effective. In the absence of expert opinion, we restrict ourselves to uniform priors on the parameters (see

Figure 2.11(a)). But in this case, we translate the inputs from the system engineers to skew priors for the two age reduction factors and retain the uniform prior for the two failure rate parameters (see Figure 2.11(b)). In the following, we compare the effect of selecting two different sets of priors. We fix $\boldsymbol{\theta}^{\mathrm{lb}} = (1, 0.001, 0, 0)$ and

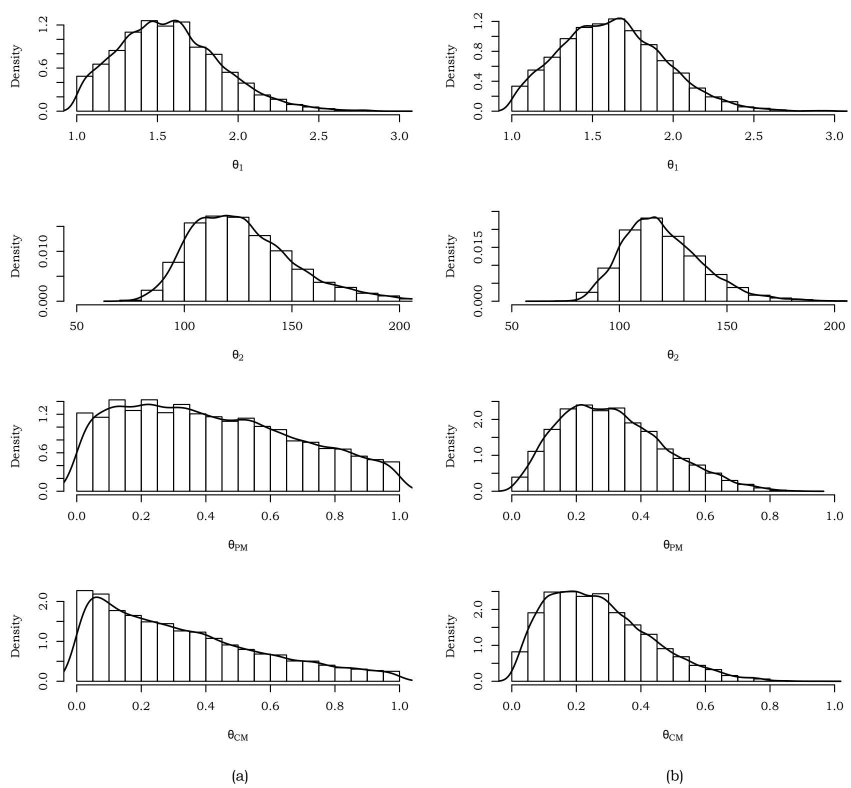

Figure 2.12: Posterior density estimates of the parameters under two sets of priors.

$\boldsymbol{\theta}^{\mathrm{ub}} = (5, 500, 1, 1)$, and run 25,000 iterations of the Gibbs sampler with $\boldsymbol{\theta}^{\mathbf{0}} = (2.5, 250, 0.5, 0.5)$. Figure 2.12 displays marginal posterior density estimates for each of the parameters. A uniform prior on θ_{CM} results in a right-skewed posterior, confirming that the data support the system engineers' belief in θ_{CM}. But it is also clear that the dataset does not provide strong evidence for the effectiveness of PMs.

Furthermore, we note that the two sets of priors for the age reduction factors did not have a significant impact on the posterior densities of the failure rate parameters. Trace plots in Figure 2.13 support good mixing and exploration by the Markov chains. Figure 2.14 suggests convergence of the Gibbs sampler. Finally, Table 2.4 presents a comparison of summary statistics and quantiles of the marginal posterior distributions, obtained using two the sets of priors.

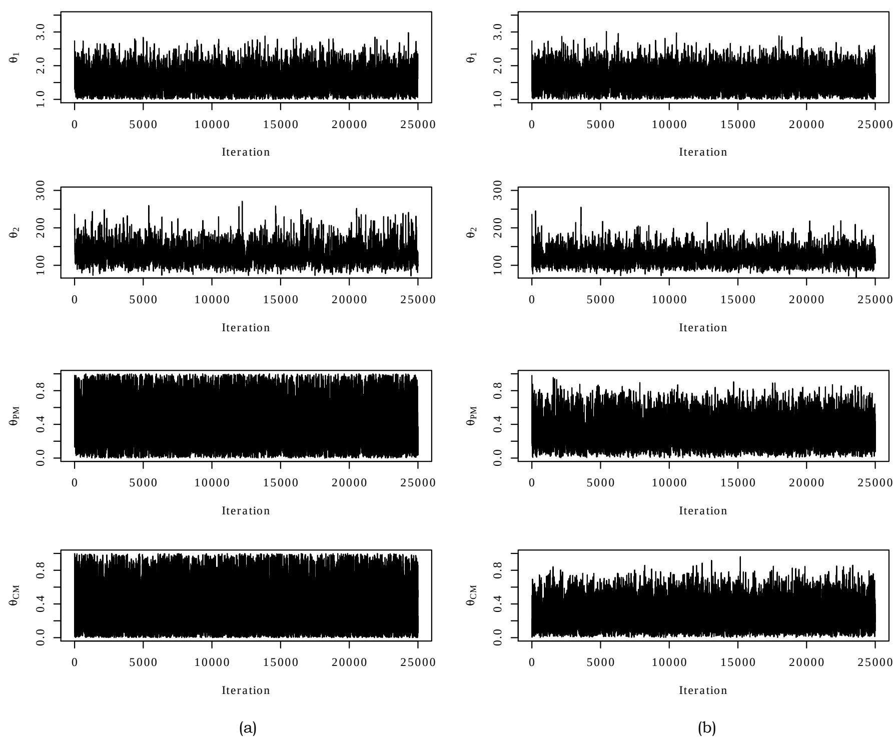

Figure 2.13: Trace plots of simulated parameters in Gibbs sampling for two sets of priors.

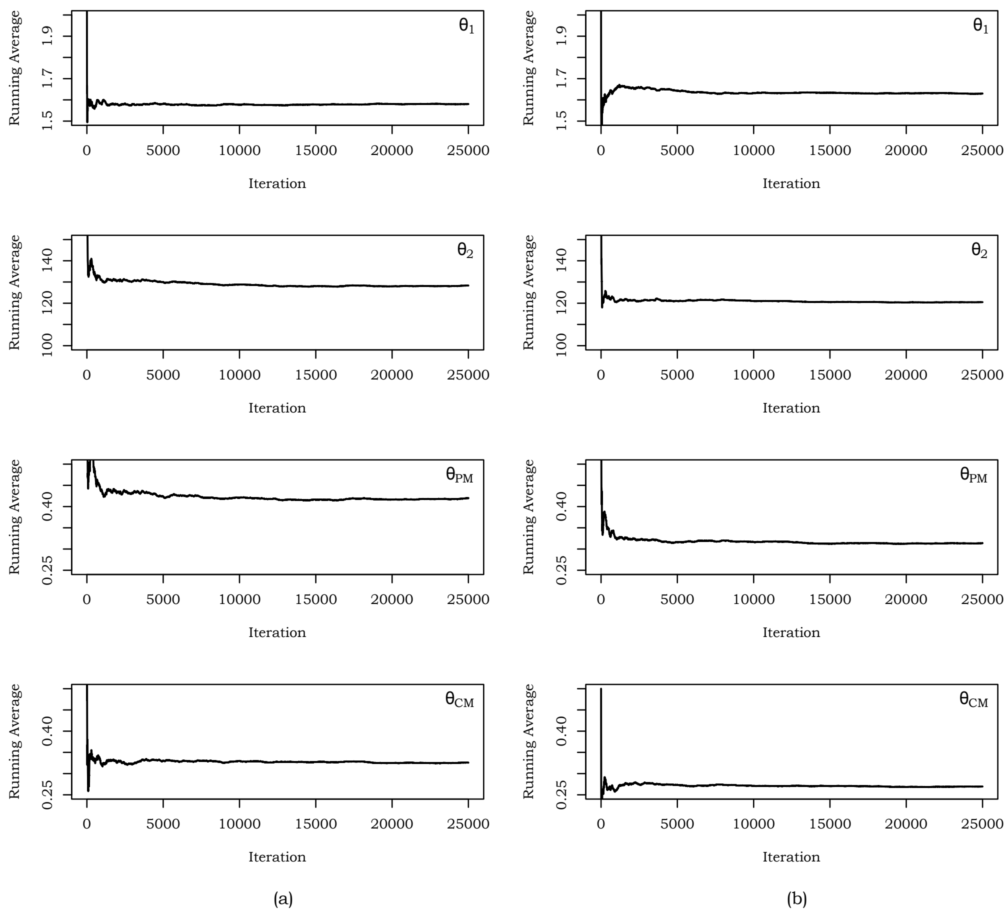

Figure 2.14: Running averages of simulated parameters for two sets of priors.

	(a)							(b)						
	Statistics		Quantiles					Statistics		Quantiles				
	Mean	SD	2.5%	25%	50%	75%	97.5%	Mean	SD	2.5%	25%	50%	75%	97.5%
θ_1	1.580	0.317	1.056	1.346	1.557	1.780	2.275	1.627	0.318	1.077	1.399	1.613	1.834	2.302
θ_2	127.92	25.287	90.20	109.32	124.24	141.79	188.31	120.38	18.78	89.87	107.12	117.79	131.40	163.88
θ_{PM}	0.419	0.264	0.021	0.198	0.388	0.617	0.946	0.313	0.162	0.058	0.189	0.296	0.417	0.667
θ_{CM}	0.325	0.249	0.009	0.115	0.271	0.483	0.903	0.269	0.156	0.038	0.146	0.247	0.366	0.624

Table 2.4: Comparison of summary statistics and quantiles of the marginal posterior distributions for two sets of priors.

Chapter 3

Simulation-Based Periodic Preventive Maintenance Optimization

In practice, a single piece of equipment may undergo different types of PM, where each PM is designed to address one or more distinct failure modes associated with the equipment. Let $\mathcal{P}$ denote the set of PM types and let $\mathcal{F}$ be the set of distinct failure modes of the equipment. It is possible that one or more failure modes in $\mathcal{F}$ are not addressed by any PM in $\mathcal{P}$, but we can re-define the set $\mathcal{F}$ to include only the failure modes accounted for by the set of PMs, $\mathcal{P}$. In general, there may exist a *many-to-many* mapping between the elements of $\mathcal{P}$ and $\mathcal{F}$. But we assume that the optimal maintenance policy of a PM in $\mathcal{P}$ is primarily governed by a unique element of $\mathcal{F}$. Therefore, we consider a simple case in which each element in $\mathcal{F}$ maps uniquely to a single element in $\mathcal{P}$ and vice-versa. In other words, we focus on the case in which there is *one-to-one* mapping between elements of $\mathcal{P}$ and elements of $\mathcal{F}$. We enrich the definition of *equipment* to represent a unique equipment-PM combination, and we begin with a maintenance optimization problem for a single piece of equipment.

3.1 Periodic PM Optimization Problem

In this chapter, we focus on a *periodic PM policy*, in which PMs are performed at times $T, 2T, \ldots, nT$ over a finite planning horizon L such that $nT \leq L$ (for some $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers). Under the policy, all intervening operational failures of the equipment are repaired through CMs. We note that a *periodic PM policy* is a special case of a *sequential PM policy*, in which PMs are carried out at times $T_1 \leq T_2 \leq \cdots \leq T_n (T_n \leq L)$. A sequential PM policy makes sense especially when the quality of the maintenance action is uncertain or at least is not modeled using the *idealized view* on CM and PM. But sequential PM policies are difficult to express. In addition, scheduling such policies for a piece of equipment, in a system of different types of equipment, can be difficult. On the other hand, a periodic PM policy is simple to express and easy to schedule, and due to these reasons, such policies are widely used in industry including NPPs.

Most of the work in the reliability literature defines a *periodic PM policy* under the assumption that $nT = L$ (see Nakagawa & Mizutani 2009, for more details). In this case, either n or T can be used identify the policy. However, a few researchers formulate and study the problem, only in terms of decision variable T (or equivalently in terms of n) and do not assume $nT = L$ in their models. But it turns out that under the *idealized view* on PMs and CMs, an optimal periodic PM policy satisfies the condition $nT = L$ (for some $n \in \mathbb{N}$; see Boland & Proschan 1982, Galenko et al. 2005). We modify the maintenance models presented in Boland & Proschan (1982) and Galenko et al. (2005) to include imperfect PMs and CMs, and use decision variable T to define the finite horizon periodic PM policy. A general two-stage periodic PM policy is defined and studied in the next chapter. Since we model imperfect maintenance actions through age reduction factors and use the concept of virtual age, it becomes equally important to mention the *age-reduction model* assumed to define the PM policy. In what follows, we assume the *Kijima-II* age-reduction model, unless we explicitly state otherwise.

For a given PM policy, a key element in a PM optimization model is the estimation of the number of equipment failures over the planning horizon. Let random variable $\mathcal{N}(L;T)$ denote the number of equipment failures on the time interval $[0,L]$ under a periodic PM policy, identified by T. Note that we have changed the notation of the number of failures from $\mathcal{N}(t)$ to $\mathcal{N}(t;T)$ to indicate dependence on T. The optimal periodic maintenance policy aims at minimizing the expected value of a random total cost function $\mathcal{C}(L;T)$. We write $\mathcal{C}(L;T)$ as sum of two random cost functions, $\mathcal{C}_p(L;T)$ and $\mathcal{C}_c(L;T)$. The former denotes the random cost of all the PMs and the latter defines the random CM cost of $\mathcal{N}(L;T)$ failures over L, for a periodic PM policy identified by T. Mathematically,

$$
\begin{aligned}
z^*(L) &= \min_{T\in[0,L]} \mathbb{E}\left[\mathcal{C}\left(L;T\right)\right] \qquad \mathcal{M}(1)\\
&= \min_{T\in[0,L]} \mathbb{E}\left[\mathcal{C}_p(L;T)+\mathcal{C}_c(L;T)\right]\\
&= \min_{T\in[0,L]} \left\{\mathbb{E}\left[\mathcal{C}_p(L;T)\right]+\mathbb{E}\left[\mathcal{C}_c(L;T)\right]\right\},
\end{aligned}
$$

where we use $\mathcal{M}(1)$ to denote our first class of periodic PM optimization models.

Galenko et al. (2005) assume that the equipment undergo PM at the end of the planning horizon L, which is a reasonable assumption in a NPP setting. The authors consider a fixed cost, C_{PM}, for each PM. Therefore, the PM cost function in their case is not random and is defined as:

$$\mathcal{C}_p(L;T) = \left\lceil \frac{L}{T} \right\rceil C_{\text{PM}}. \tag{3.1}$$

We note that $\lceil\cdot\rceil$ in equation (3.1) is the ceiling operator and $\left\lceil \frac{L}{T} \right\rceil$ denotes the total number of PMs over L, when a PM must be performed at L. Furthermore, the authors assume that an operational failure of a piece of equipment can result in a fixed downtime cost, C_{D} with probability p. If C^o_{CM} and C_{CM} denote the fixed and expected cost associated with a CM then $C_{\text{CM}} = pC_{\text{D}} + (1-p)C^o_{\text{CM}}$. The total CM cost function in their work is random only because it is defined over a random number of equipment failures, and can be written as:

$$\mathcal{C}_c(L;T) = \mathcal{N}(L;T)\, C_{\text{CM}}.$$

Let $R(t) = \displaystyle\int_0^t r(u)\mathrm{d}u$, where $r(\cdot)$ is the failure rate associated with the distribution of the time to first equipment failure. Under the *idealized view* on PMs and CMs, the number of failures within a PM interval of length T follows Poisson distribution with mean $R(T)$ (see Barlow & Hunter 1960). When decision variable T is used to define a periodic PM policy then there are exactly $\left\lfloor \dfrac{L}{T} \right\rfloor$ PM intervals of length T. In case $nT \neq L$ for some $n \in \mathbb{N}$, there is an additional interval of length $L - \left\lfloor \dfrac{L}{T} \right\rfloor T$. Therefore, random variable $\mathcal{N}(L;T)$ is a sum of exactly $\left\lceil \dfrac{L}{T} \right\rceil$ independent Poisson random variables, the first $\left\lfloor \dfrac{L}{T} \right\rfloor$ of which have a Poisson distribution with mean $R(T)$. When $nT \neq L$, the number of failures in the last PM interval follows a Poisson distribution with mean $R\left(L - \left\lfloor \dfrac{L}{T} \right\rfloor T\right)$. Thus, $\mathcal{N}(L;T)$ is Poisson random variable with mean, $\Lambda(L;T) = \mathbb{E}\left[\mathcal{N}(L;T)\right]$, where

$$\Lambda(L;T) = \left\lfloor \frac{L}{T} \right\rfloor R\left(T\right) + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right). \tag{3.2}$$

Equation (3.2) defines the expected number of equipment failures on $[0, L]$. Here, $\Lambda(L;T)$ is a sum of the expected number of failures, over all PM intervals. Let $\Lambda_2(L;T)$ be the second moment of random variable $\mathcal{N}(L;T)$. Then,

$$\Lambda_2(L;T) = \mathbb{E}\left[\mathcal{N}(L;T)^2\right] = \Lambda(L;T)\left(\Lambda(L;T) + 1\right). \tag{3.3}$$

Note that

$$\Lambda_2(L;T) \geq \left\lfloor \frac{L}{T} \right\rfloor [R(T)]\,[R(T)+1] + \left[R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right)\right] \left[R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right) + 1\right], \tag{3.4}$$

where equality holds when $T = L$.

Galenko et al. (2005) assume perfect PMs with minimal repairs (CMs) and solve the following PM optimization problem:

$$\begin{aligned}
z_1^*(L) &= \min_{T\in[0,L]} \mathbb{E}\left[\mathcal{C}\left(L;T\right)\right] \\
&= \min_{T\in[0,L]} \left\{\mathbb{E}\left[\mathcal{C}_p(L;T)\right] + \mathbb{E}\left[\mathcal{C}_c(L;T)\right]\right\} \\
&= \min_{T\in[0,L]} \left\{\mathbb{E}\left(\left\lceil \frac{L}{T} \right\rceil C_{\mathrm{PM}}\right) + \mathbb{E}\left[\mathcal{N}(L;T)C_{\mathrm{CM}}\right]\right\} \\
&= \min_{T\in[0,L]} \left\{\left\lceil \frac{L}{T} \right\rceil C_{\mathrm{PM}} + \Lambda(L;T)C_{\mathrm{CM}}\right\} \\
&= \min_{T\in[0,L]} \left\{\left\lceil \frac{L}{T} \right\rceil C_{\mathrm{PM}} + \left(\left\lfloor \frac{L}{T} \right\rfloor R\left(T\right) + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right)\right) C_{\mathrm{CM}}\right\}.
\end{aligned} \tag{3.5}$$

We note that the maintenance model presented in (3.5) is a *risk neutral* version of $\mathcal{M}(1)$ since it only considers the expected number of failures.

For NPPs frequent equipment failures are highly undesirable. Therefore, we seek a *risk-averse* version of $\mathcal{M}(1)$, which penalizes subsequent equipment failures at a rate higher than C_{CM}. In this direction, Boland & Proschan (1982) suggest one such type of a cost function in which the j^{th} equipment failure *within a PM interval* costs $C_{\mathrm{CM}} + j\ C_{\Delta}$, where C_{Δ} is the incremental cost of a CM. If $C_{\mathrm{CM}}(m)$ denotes the total CM cost of m failures within a PM interval then

$$C_{\mathrm{CM}}(m) = \sum_{j=1}^{m} \left(C_{\mathrm{CM}} + j\ C_{\Delta}\right) = mC_{\mathrm{CM}} + \frac{m}{2}(m+1)C_{\Delta}. \tag{3.6}$$

Therefore, the total expected cost of CMs over $[0, L]$ under a periodic PM policy, identified by T can be written as:

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{C}_c(L;T)\right] = &\left\lfloor \frac{L}{T} \right\rfloor \sum_{m=1}^{\infty} \left\{ \frac{\exp\left[-R(T)\right]\left[R(T)\right]^m}{m!} \left(mC_{\text{CM}} + \frac{m}{2}(m+1)C_{\Delta} \right) \right\} \\
&+ \sum_{m=1}^{\infty} \left\{ \frac{\exp\left[-R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right)\right]\left[R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right)\right]^m}{m!} \left(mC_{\text{CM}} + \frac{m}{2}(m+1)C_{\Delta} \right) \right\} \\
= &\left\lfloor \frac{L}{T} \right\rfloor \left\{ R(T)C_{\text{CM}} + \left[R(T)\left(R(T)+1\right) + R(T)\right] \frac{C_{\Delta}}{2} \right\} + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) C_{\text{CM}} \\
&+ \left[R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) \left(R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) + 1 \right) + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) \right] \frac{C_{\Delta}}{2} \\
= &\left\{ \left\lfloor \frac{L}{T} \right\rfloor R(T) + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) \right\} \left(C_{\text{CM}} + C_{\Delta} \right) \\
&+ \left\{ \left\lfloor \frac{L}{T} \right\rfloor \left[R(T)\right]^2 + \left[R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) \right]^2 \right\} \frac{C_{\Delta}}{2}.
\end{aligned}
$$

Since Boland & Proschan (1982) do not assume that a PM is performed at L, an equivalent formulation of their maintenance optimization model is,

$$
\begin{aligned}
z_2^*(L) &= \min_{T \in [0,L]} \mathbb{E}\left[\mathcal{C}\left(L;T\right)\right] \\
&= \min_{T \in [0,L]} \left\{ \mathbb{E}\left[\mathcal{C}_p(L;T)\right] + \mathbb{E}\left[\mathcal{C}_c(L;T)\right] \right\} \\
&= \min_{T \in [0,L]} \left\{ \left(\left\lceil \frac{L}{T} \right\rceil - 1 \right) C_{\text{PM}} + \left(\left\lfloor \frac{L}{T} \right\rfloor R(T) + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) \right) \left(C_{\text{CM}} + C_{\Delta} \right) \right. \\
&\qquad \left. + \left(\left\lfloor \frac{L}{T} \right\rfloor \left[R(T)\right]^2 + \left[R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T \right) \right]^2 \right) \frac{C_{\Delta}}{2} \right\}. \qquad (3.7)
\end{aligned}
$$

Boland & Proschan (1982) and Galenko et al. (2005) assume the *idealized view* on PMs and CMs, which leads to an analytical expression for $\Lambda(L;T)$ (see equation (3.2)). Under the assumption of an increasing failure rate function, $r(\cdot)$, Boland & Proschan (1982) show that $\mathbb{E}\left[\mathcal{C}\left(L;T\right)\right]$ given in equation (3.7) is right continuous in T, with possible points of discontinuity in the set $\mathbb{D} = \left\{ d : d = \frac{L}{n}, n \in \mathbb{N} \right\}$. The authors prove that $\mathbb{E}\left[\mathcal{C}\left(L;T\right)\right]$ is minimized at one of the points in the set $\mathbb{D}$.

Galenko et al. (2005) consider a special case of the Boland & Proschan (1982) model. Note that if we set $C_{\Delta} = 0$ and assume that a PM is always performed at L, equation (3.7) reduces to (3.5). Galenko et al. (2005) show that $\mathbb{E}\left[\mathcal{C}\left(L;T\right)\right]$ given in equation (3.5) is lower semi-continuous in T with $\mathbb{D}$ as the set of discontinuities and is increasing and convex on each interval $\left[\frac{L}{n}, \frac{L}{n+1}\right)$ for $n \in \mathbb{N}$. The authors further establish that $\mathbb{E}\left[\mathcal{C}_r\left(L;T\right)\right]$ is quasi-convex on $[0, L]$, where $\mathbb{E}\left[\mathcal{C}_r\left(L;T\right)\right] = \dfrac{L}{T}\left(C_{\text{PM}} + C_{\text{CM}} R(T)\right)$. If T_r^* minimizes $\mathbb{E}\left[\mathcal{C}_r\left(L;T\right)\right]$ and T^* solves (3.5) then Galenko et al. (2005) prove that $T^* \in \arg\min_{T \in \left\{\frac{L}{n^*+1}, \frac{L}{n^*}\right\}} \mathbb{E}\left[\mathcal{C}\left(L;T\right)\right]$ such that $\dfrac{L}{n^*+1} \leq T_r^* \leq \dfrac{L}{n^*}$. Finally, Galenko et al. (2005) present an efficient algorithm to solve this single equipment maintenance optimization problem.

3.1.1 An Alternative Risk-Averse Periodic PM Optimization Model

Boland & Proschan (1982) characterize the risk associated with a CM through a cost function defined over a given PM interval (see equation (3.6)). The model then aggregates the total CM cost over all possible PM intervals up to L, for a periodic PM policy identified through T. The authors implicitly assume that a PM resets the risk associated with a failure. For example, every j^{th} CM from different PM intervals has equal cost. Furthermore, the j^{th} CM of a given PM interval costs less than the $j'^{\text{ th}}$ CM from the same or any other PM interval, $\forall j' < j$ and $C_{\Delta} > 0$. For simplicity, we say that such a model uses a *localized* notion of risk since the CM cost function is defined at the PM-interval level.

For NPPs, it is reasonable to associate high CM cost with subsequent equipment failures on $[0, L]$, independent of the PM interval from which the failures are observed. Therefore, we extend the idea presented in Boland & Proschan (1982) and identify the risk associated with a CM through a cost function, defined over the length of the maintenance planning horizon, L. Let $\mathcal{C}_{\text{CM}}$ be the random cost of performing a CM, assumed independent of $\mathcal{N}(L;T)$, and $C_{\text{CM}} = \mathbb{E}\left[\mathcal{C}_{\text{CM}}\right] < \infty$. We assume that j^{th} equipment failure on $[0, L]$ costs $\mathcal{C}_{\text{CM}} + j\ \mathcal{C}_{\Delta}$, where $\mathcal{C}_{\Delta}$ is the random incremental cost, associated with a CM. For convenience, we assume $\mathcal{C}_{\Delta} = \alpha\ \mathcal{C}_{\text{CM}}$, for some $\alpha > 0$, where the value of α is primarily governed by the criticality of the equipment failure. STP, for example, maintains a risk rank associated with all the important equipment failure mode combinations, which can be directly used to select the value of α. We use $\mathcal{C}_{\text{CM}}(m)$ to denote the total random cost of m failures on $[0, L]$, where the j^{th} equipment failure costs $\mathcal{C}_{\text{CM}} + \alpha\ j\ \mathcal{C}_{\text{CM}}$. For a given value of m, we write,

$$\mathcal{C}_{\text{CM}}(m) = \sum_{j=1}^{m} \left(\mathcal{C}_{\text{CM}} + \alpha\ j\ \mathcal{C}_{\text{CM}}\right) = \left(\left(1 + \frac{\alpha}{2}\right) m + \frac{\alpha}{2} m^2\right) \mathcal{C}_{\text{CM}}. \tag{3.8}$$

The CM cost function $\mathcal{C}_{\text{CM}}(\cdot)$ given in equation (3.8) is defined over the total number of equipment failures in $[0, L]$. For simplicity, we say that $\mathcal{C}_{\text{CM}}(\cdot)$ captures the risk at a *global* level, to differentiate from the *localized* notion of risk used in Boland & Proschan (1982).

Let $\mathcal{C}_{\text{PM}}$ be the random cost of performing a PM and $C_{\text{PM}} = \mathbb{E}\left[\mathcal{C}_{\text{PM}}\right] < \infty$ be its expected value. We assume that the equipment cannot have an infinite number of failures over the finite planning horizon, L. Therefore, $\Lambda(L;T) = \mathbb{E}\left[\mathcal{N}(L;T)\right] < \infty$ and $\Lambda_2(L;T) = \mathbb{E}\left[\mathcal{N}(L;T)^2\right] < \infty$. An alternative risk-averse formulation of $\mathcal{M}(1)$ is:

$$\begin{aligned} z_3^*(L) &= \min_{T \in [0,L]} \mathbb{E}\left[\mathcal{C}\left(L;T\right)\right] \\ &= \min_{T \in [0,L]} \left\{\mathbb{E}\left[\mathcal{C}_p(L;T)\right] + \mathbb{E}\left[\mathcal{C}_c(L;T)\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \min_{T\in[0,L]} \left\{\mathbb{E}\left[\mathcal{C}_p(L;T)\right] + \mathbb{E}\left[\mathcal{C}_{\mathrm{CM}}\left(\mathcal{N}\left(L;T\right)\right)\right]\right\} \\
&= \min_{T\in[0,L]} \left\{\mathbb{E}\left[\mathcal{C}_p(L;T)\right] + \mathbb{E}\left[\left(\left(1+\frac{\alpha}{2}\right)\mathcal{N}(L;T) + \frac{\alpha}{2}\mathcal{N}(L;T)^2\right)\mathcal{C}_{\mathrm{CM}}\right]\right\} \\
&= \min_{T\in[0,L]} \left\{\mathbb{E}\left[\mathcal{C}_p(L;T)\right] + \left(\left(1+\frac{\alpha}{2}\right)\Lambda(L;T) + \frac{\alpha}{2}\Lambda_2(L;T)\right)C_{\mathrm{CM}}\right\}. \qquad (3.9)
\end{aligned}
$$

If a PM must be performed at L then

$$
\mathbb{E}\left[\mathcal{C}_p(L;T)\right] = \mathbb{E}\left[\left\lceil\frac{L}{T}\right\rceil\mathcal{C}_{\mathrm{PM}}\right] = \left\lceil\frac{L}{T}\right\rceil C_{\mathrm{PM}},
$$

and otherwise,

$$
\mathbb{E}\left[\mathcal{C}_p(L;T)\right] = \mathbb{E}\left[\left(\left\lceil\frac{L}{T}\right\rceil - 1\right)\mathcal{C}_{\mathrm{PM}}\right] = \left(\left\lceil\frac{L}{T}\right\rceil - 1\right)C_{\mathrm{PM}}.
$$

We note that setting $\alpha = 0$ in (3.9) results in a *risk-neutral* formulation of $\mathcal{M}(1)$.

3.2 Simulating Number of Failures

Analytical solution to the periodic PM optimization models presented in Section 3.1 requires closed-form expressions for $\Lambda(L;T)$ and $\Lambda_2(L;T)$. Unfortunately, such expressions are only available when CMs are modeled as minimal repairs (PMs can be imperfect). For general values of the CM age reduction factor θ_{CM}, $\Lambda(L;T)$ and $\Lambda_2(L;T)$ must be estimated by simulation. Towards this end, we describe a procedure from Jack (1998) to simulate $\mathcal{N}(L;T)$, for a fixed value of $\boldsymbol{\theta}$. Recall that $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q, \theta_{\mathrm{PM}}, \theta_{\mathrm{CM}})$, where $(\theta_1, \theta_2, \ldots, \theta_q)$ are parameters of the failure rate function $r(\cdot)$ and $R(t) = \int_0^t r(u)\mathrm{d}u$. As in Section 2.2, we let random variables $T_{0(j+1)}$ and T_{ij} ($j = 1, 2, \ldots, k,\ i = 1, 2, \ldots, n_j$) denote the time of the j^{th} PM and the i^{th} CM in the j^{th} PM interval, respectively. We use random variable $X_{0(j+1)} = T_{0(j+1)} - T_{n_j j}$ to represent the time between the j^{th} PM and the ${n_j}^{\text{th}}$ maintenance (CM or PM) in the j^{th} PM interval. Similarly, random variable $X_{ij} = T_{ij} - T_{(i-1)j}$ indicates the time between the i^{th} CM and the $(i-1)^{\text{st}}$ maintenance (CM or PM) in the j^{th} PM interval. See Figure 3.1.

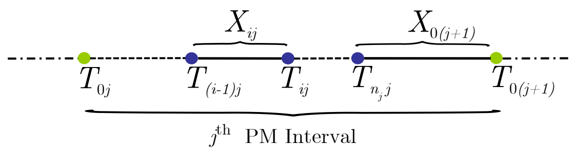

Figure 3.1: Random variables in the j^{th} PM interval.

Let $\mathbf{t}_{(i-1)j} = \left(t_{11}, t_{21}, \ldots, t_{(i-1)j}\right)$ be generic realizations of random variables $\left(T_{11}, T_{21}, \ldots, T_{(i-1)j}\right)$. We note that for a given age-reduction model, a fixed value of $\boldsymbol{\theta}$ together with $\mathbf{t}_{(i-1)j}$ represents the virtual age following the $(i-1)^{\text{st}}$ maintenance (CM or PM) in the j^{th} PM interval, $v_{(i-1)j}$. Furthermore, virtual age $v_{(i-1)j}$ at calendar time $t_{(i-1)j}$ represents new equipment that has survived $v_{(i-1)j}$ time units. Therefore, we can write the conditional survival function for X_{ij} as,

$$
\begin{aligned}
\mathbb{P}\left\{X_{ij} > x \mid \mathbf{t}_{(i-1)j},\ \boldsymbol{\theta}\right\} &= \mathbb{P}\left\{X_{ij} > x \mid v_{(i-1)j}\right\}. \\
&= \mathbb{P}\left\{X_{11} > x + v_{(i-1)j} \mid X_{11} > v_{(i-1)j}\right\} \\
&= \frac{\mathbb{P}\left\{X_{11} > x + v_{(i-1)j} \text{ and } X_{11} > v_{(i-1)j}\right\}}{\mathbb{P}\left\{X_{11} > v_{(i-1)j}\right\}} \\
&= \frac{\mathbb{P}\left\{X_{11} > x + v_{(i-1)j}\right\}}{\mathbb{P}\left\{X_{11} > v_{(i-1)j}\right\}} \\
&= \frac{\exp\left(-\displaystyle\int_0^{x+v_{(i-1)j}} r(u)\mathrm{d}u\right)}{\exp\left(-\displaystyle\int_0^{v_{(i-1)j}} r(u)\mathrm{d}u\right)} \\
&= \exp\left(-R(x + v_{(i-1)j}) + R(v_{(i-1)j})\right).
\end{aligned}
$$

We use an inverse transformation to generate a random variable from the conditional survival function. Let $U(0,1)$ be a standard uniform random variable. Then,

$$\begin{aligned}
& \exp\left(-R(X+v_{(i-1)j})+R(v_{(i-1)j})\right)=U(0,1), \\
\Rightarrow \qquad & R(X+v_{(i-1)j})=R(v_{(i-1)j})-\ln U(0,1), \\
\Rightarrow \qquad & X=-v_{(i-1)j}+R^{-1}\left(R(v_{(i-1)j})-\ln U(0,1)\right). \qquad (3.10)
\end{aligned}$$

In the last step (equation (3.10)), we have assumed that $R(\cdot)$ is invertible. For a periodic PM policy defined by T over $[0,L]$, we set $T_{ij}=\min\left(L,\ jT,\ t_{(i-1)j}+X\right)$, where $T_{ij}=jT$ indicates a PM and $T_{ij}=t_{(i-1)j}+X$ represents a failure. When $T_{ij}=L$, a sample path simulating the failure process on $[0,L]$ is complete. We estimate $\Lambda(L;T)$ and $\Lambda_2(L;T)$ by simulating independent sample paths terminating at L. Let $\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T)$ denote the number of failures in the m^{th} simulated sample path, for a fixed value of $\boldsymbol{\theta}$. If M independent sample paths are generated then for a fixed value of $\boldsymbol{\theta}$,

$$\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)=\frac{1}{M}\sum_{m=1}^{M}\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T), \qquad (3.11)$$

provides an estimate for $\Lambda_{\boldsymbol{\theta}}(L;T)$, where $\Lambda_{\boldsymbol{\theta}}(L;T)=\mathbb{E}\left[\mathcal{N}(L;T)\mid\boldsymbol{\theta}\right]$. Similarly,

$$\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T)=\frac{1}{M}\sum_{m=1}^{M}\left(\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T)\right)^2, \qquad (3.12)$$

estimates $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$, where $\Lambda_{(2,\boldsymbol{\theta})}(L;T)=\mathbb{E}\left[\mathcal{N}(L;T)^2\mid\boldsymbol{\theta}\right]$. Algorithm 1 outlines our procedure to generate M independent sample paths described above and reports estimates for $\Lambda_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$ under *Kijima-II*. We note that lines 9 and 13 in Algorithm 1 can be easily modified to include other age-reduction models.

Algorithm 1 Estimation of $\Lambda_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$ for a fixed $\boldsymbol{\theta}$

Input: $L,\ T,\ \boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q, \theta_{\text{PM}}, \theta_{\text{CM}}),\ R(\cdot),\ R^{-1}(\cdot),\ M,\ v_0$

Output: $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T),\ \hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T)$

for $m = 1, 2, \ldots, M$ **do**
  $\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T) \leftarrow 0; \quad j \leftarrow 1; \quad v \leftarrow v_0; \quad t \leftarrow 0; \quad t' \leftarrow 0;$
  while $(t < L)$ **do**
    $U \leftarrow U(0,1)$
    $x \leftarrow -v + R^{-1}\left[R(v) - \ln U\right]$
    $t' \leftarrow t + x$
    if $(t' < jT$ AND $t' < L)$ **then**
      $\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T) \leftarrow \hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T) + 1$
      $v \leftarrow \theta_{\text{CM}}\,(v + x)$
      $t \leftarrow t'$
    else
      if $(jT \leq L)$ **then**
        $v \leftarrow \theta_{\text{PM}}\,(v + jT - t)$
        $t \leftarrow jT$
        $j \leftarrow j + 1$
      else
        $v \leftarrow (v + L - t)$
        $t \leftarrow L$
      end if
    end if
  end while
end for
$\hat{\Lambda}_{\boldsymbol{\theta}}(L;T) \leftarrow \frac{1}{M}\sum_{m=1}^{M} \hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T)$
$\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T) \leftarrow \frac{1}{M}\sum_{m=1}^{M} \left(\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T)\right)^2$

3.2.1 Simulation Experiences

Perfect PMs and Minimal CMs

By the strong law of large numbers, $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T) = \frac{1}{M}\sum_{m=1}^{M} \hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T) \rightarrow \Lambda_{\boldsymbol{\theta}}(L;T)$ and $\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T) = \frac{1}{M}\sum_{m=1}^{M}\left(\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T)\right)^2 \rightarrow \Lambda_{(2,\boldsymbol{\theta})}(L;T)$, with probability 1, as $M \rightarrow \infty$. We study the quality of these estimates, $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ and $\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T)$, for different values of M. In this direction, we assume a Weibull distribution for X_{11} and fix $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_{\text{PM}}, \theta_{\text{CM}}) = (2, 20, 0, 1)$, where θ_1 is the *shape* and θ_2 is the *scale* parameter of the Weibull distribution. Recall that we assume the *Kijima-II* age-reduction model, unless stated otherwise. For perfect PMs and minimal CMs, equations (3.2) and (3.3) provide closed-form expressions for $\Lambda_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$, respectively. Note that when $\boldsymbol{\Theta} = \boldsymbol{\theta}$ with probability 1, $\Lambda(L;T) = \Lambda_{\boldsymbol{\theta}}(L;T)$ and similarly, $\Lambda_2(L;T) = \Lambda_{(2,\boldsymbol{\theta})}(L;T)$.

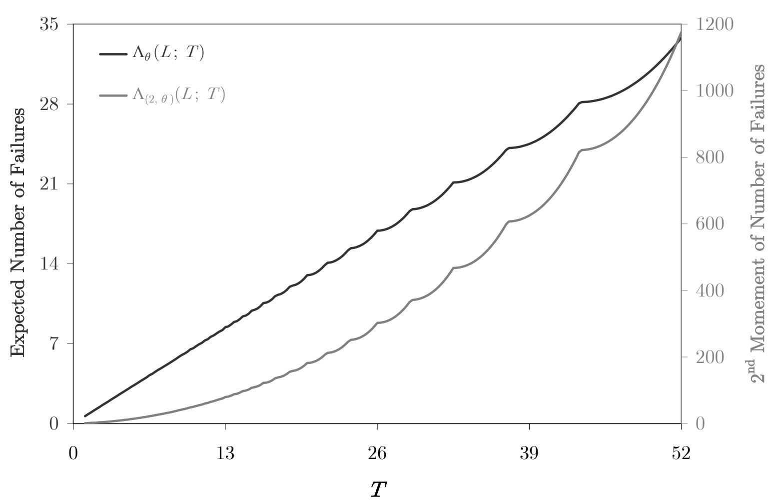

Figure 3.2: Theoretical values of $\Lambda_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$ with perfect PMs and minimal CMs ($\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_{\text{PM}}, \theta_{\text{CM}}) = (2, 20, 0, 1)$ and $L = 260$).

Figure 3.2 plots $\Lambda_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$, as a function of T when $L = 260$ (note the different scales on the y-axes). Figure 3.3 compares (i) $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ and

$\Lambda_{\boldsymbol{\theta}}(L;T)$, and (ii) $\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$, for different values of M. For a fixed value of M, we call Algorithm 1 to generate $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ and $\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T)$, using different values of T. We use solid lines, and circles on dotted lines to indicate theoretical and simulated values, respectively, where theoretical values are obtained from the closed-form expressions.

We first validate that for small values of M (e.g., $M = 10$), $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ and $\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T)$ do not provide reasonable estimates for $\Lambda_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$, respectively (see Figure 3.3(a)). For moderate values of M (e.g., $M = 100$), the estimators provide reasonably good approximations (see Figure 3.3(b)). Finally, for large M (e.g., $M = 500$ and $M = 1000$), $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ and $\hat{\Lambda}_{(2,\boldsymbol{\theta})}(L;T)$ result in very accurate estimates (see Figures 3.3(c) and 3.3(d)).

Imperfect PMs and Minimal CMs

As mentioned earlier, it is also possible to write a closed-form expression for $\Lambda_{\boldsymbol{\theta}}(L;T)$ when PMs are imperfect and CMs are modeled as minimal repairs. Forming the geometric sum, let

$$\breve{\theta}_{\text{PM}}^{(n)} = \begin{cases} \displaystyle\sum_{j=1}^{n} (\theta_{\text{PM}})^j & n \in \mathbb{N} \\ 0 \quad \text{otherwise} \end{cases}, \tag{3.13}$$

and note that

$$\theta_{\text{PM}} \left(\breve{\theta}_{\text{PM}}^{(n-1)} + 1 \right) = \breve{\theta}_{\text{PM}}^{(n)} \qquad n \in \mathbb{N}. \tag{3.14}$$

If we assume $v(0) = 0$ then the virtual age of the equipment following the n^{th} PM at nT can be derived as follows:

$$v(T) = \theta_{\text{PM}} \Big(v(0) + T \Big) = \theta_{\text{PM}} T = \breve{\theta}_{\text{PM}}^{(1)} T$$

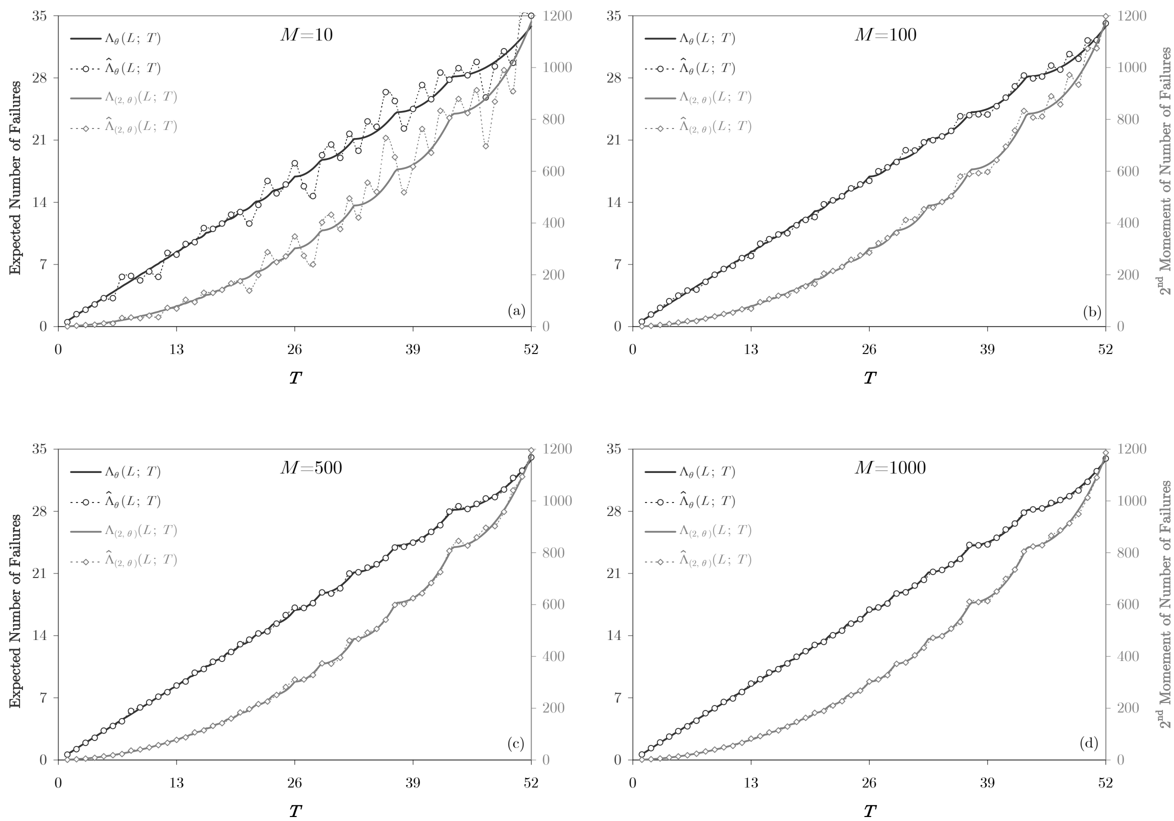

Figure 3.3: Comparison between estimated and theoretical values of $\Lambda_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta})}(L;T)$ when (a) $M = 10$, (b) $M = 100$, (c) $M = 500$, and (d) $M = 1000$ ($\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_{\text{PM}}, \theta_{\text{CM}}) = (2, 20, 0, 1)$ and $L = 260$).

$$
\begin{aligned}
v(2T) &= \theta_{\mathrm{PM}}\Big(v(T)+T\Big) = \theta_{\mathrm{PM}}\left(\breve{\theta}_{\mathrm{PM}}^{(1)}+1\right)T = \breve{\theta}_{\mathrm{PM}}^{(2)}T \\
&\vdots \\
v(nT) &= \theta_{\mathrm{PM}}\Big(v\left((n-1)T\right)+T\Big) = \theta_{\mathrm{PM}}\left(\breve{\theta}_{\mathrm{PM}}^{(n-1)}+1\right)T \\
&= \breve{\theta}_{\mathrm{PM}}^{(n)}T \quad n \in \mathbb{N}.
\end{aligned} \tag{3.15}
$$

Therefore, the expected number of equipment failures, $\Lambda(L;T)$, for a general value of θ_{PM}, is

$$
\begin{aligned}
\Lambda_{\boldsymbol{\theta}}(L;T) &= \mathbb{E}\left[\mathcal{N}(L;T) \mid \boldsymbol{\theta}\right] \\
&= \sum_{n=1}^{\lfloor \frac{L}{T} \rfloor} \Big\{ R\Big(v\left((n-1)T\right)+T\Big) - R\Big(v\left((n-1)T\right)\Big)\Big\} \\
&\quad + R\left(v\left(\left\lfloor \frac{L}{T} \right\rfloor T\right) + L - \left\lfloor \frac{L}{T} \right\rfloor T\right) - R\left(v\left(\left\lfloor \frac{L}{T} \right\rfloor T\right)\right), \\
&= \sum_{n=1}^{\lfloor \frac{L}{T} \rfloor} \left\{ R\left((\breve{\theta}_{\mathrm{PM}}^{(n-1)}+1)T\right) - R\left(\breve{\theta}_{\mathrm{PM}}^{(n-1)}T\right)\right\} \\
&\quad + R\left(\breve{\theta}_{\mathrm{PM}}^{(\lfloor \frac{L}{T} \rfloor)}T + L - \left\lfloor \frac{L}{T} \right\rfloor T\right) - R\left(\breve{\theta}_{\mathrm{PM}}^{(\lfloor \frac{L}{T} \rfloor)}T\right).
\end{aligned} \tag{3.16}
$$

Notice that if $\theta_{PM} = 0$, equation (3.16) reduces to (3.2). We use equation (3.16) and study the effect of the PM age reduction factor on the expected number of failures. Figure 3.4 plots $\Lambda_{\boldsymbol{\theta}}(L;T)$ as a function of T, for different values of θ_{PM} when $\boldsymbol{\theta} = (2, 20, \theta_{\mathrm{PM}}, 1)$ and $L = 260$. It is clear that deteriorating quality of PMs results in a higher number of equipment failures (in expectation). At $\theta_{PM} = 1$, $\Lambda_{\boldsymbol{\theta}}(L;T)$ is independent of T, since both PMs and CMs have no impact on the virtual age of the equipment. The expected number of failures in this case bounds $\Lambda_{\boldsymbol{\theta}}(L;T)$ from above, for any combination of the two age reduction factors. We set $\boldsymbol{\theta} = (2, 20, 0.5, 1)$ and once again use Algorithm 1 to obtain $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ for different values of

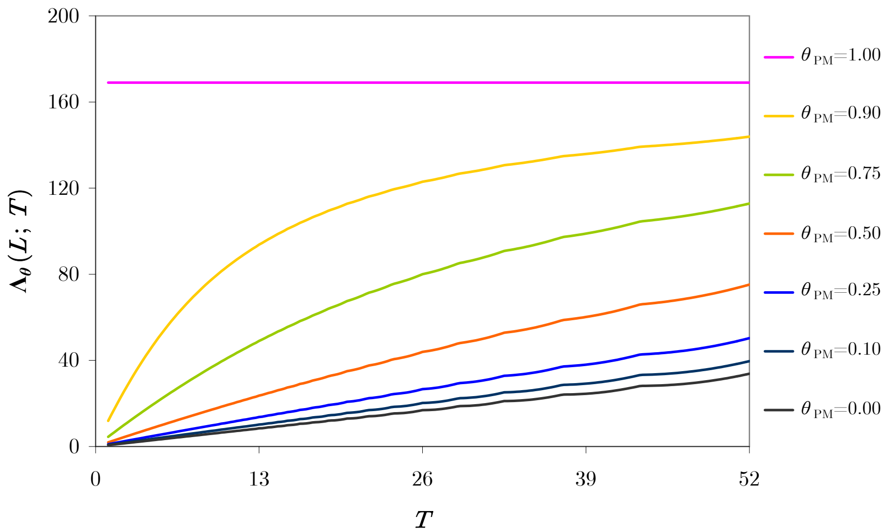

Figure 3.4: Effect of imperfect PMs on $\Lambda_{\boldsymbol{\theta}}(L;T)$ when CMs are modeled as minimal repairs ($\boldsymbol{\theta} = (2, 20, \theta_{\rm PM}, 1)$ and $L = 260$).

M. Theoretical values, $\Lambda_{\boldsymbol{\theta}}(L;T)$, are obtained using equation (3.16). A comparison between $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{\boldsymbol{\theta}}(L;T)$ is presented in Figure 3.5. The results once again indicate that $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ provides accurate estimates for $\Lambda_{\boldsymbol{\theta}}(L;T)$ for moderate to large values of M.

Perfect PMs and Imperfect CMs

For general values of the CM age reduction factor, $\theta_{\rm CM}$, it is difficult to construct a closed-form expression for the expected number of failures, $\Lambda_{\boldsymbol{\theta}}(L;T)$. In such cases, $\Lambda_{\boldsymbol{\theta}}(L;T)$ must be estimated through simulation. Comparison results presented above suggest that for large values of M, Algorithm 1 accurately estimates $\Lambda_{\boldsymbol{\theta}}(L;T)$. We next study the effect that $\theta_{\rm CM}$ has on the expected number of failures. Figure 3.6 plots $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ as a function of T for different values of the CM age reduction factor, $\theta_{\rm CM}$. The solid line indicates the case in which PMs are perfect and CMs are modeled as minimal repairs. The results clearly show that improvement

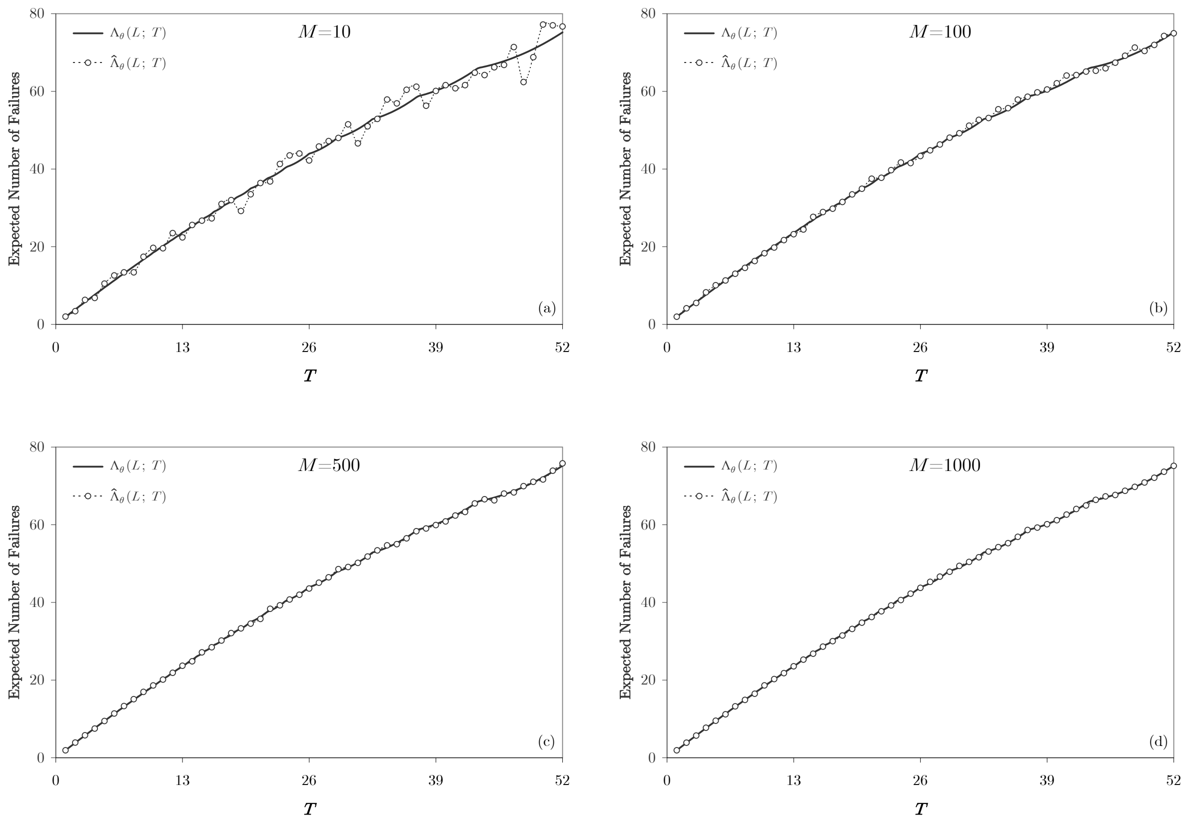

Figure 3.5: Comparison between $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ and $\Lambda_{\boldsymbol{\theta}}(L;T)$ when PMs are imperfect and CMs are modeled as minimal repairs (a) $M = 10$, (b) $M = 100$, (c) $M = 500$, and (d) $M = 1000$ ($\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_{\mathrm{PM}}, \theta_{\mathrm{CM}}) = (2, 20, 0.5, 1)$ and $L = 260$).

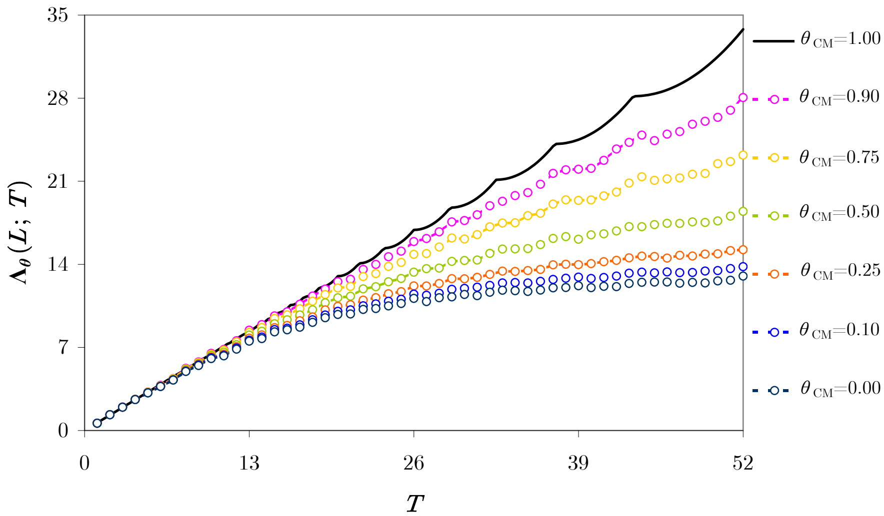

Figure 3.6: Effect of imperfect CMs on $\Lambda_{\boldsymbol{\theta}}(L;T)$ when PMs are perfect ($\boldsymbol{\theta} = (2, 20, 0, \theta_{\text{CM}})$, $L = 260$ and $M = 1000$).

in CM quality (with respect to minimal repair) results in fewer equipment failures. The expected number of failures for $\theta_{\text{CM}} = 0$, bounds $\Lambda_{\boldsymbol{\theta}}(L;T)$ from below, for any combination of the two age reduction factors.

Imperfect PMs and CMs

In the previous sections, we study the effect of one age reduction factor on the expected number of failures, when an *idealized view* is adopted to model the other factor. In practical situations, both maintenance actions, PMs and CMs, can be imperfect. Therefore, it is important to know the effect of both age reduction factors on $\Lambda_{\boldsymbol{\theta}}(L;T)$, when the factors take on general values. In the absence of a closed-form expression for $\Lambda_{\boldsymbol{\theta}}(L;T)$, we call Algorithm 1 using different values of T, and estimate $\Lambda_{\boldsymbol{\theta}}(L;T)$ by running $M = 1000$ independent simulations for every selected value of T. Figure 3.7 shows $\hat{\Lambda}_{\boldsymbol{\theta}}(L;T)$ as a function of T for different combinations of PM and CM age reduction factors. The solid line once again indicates the case when

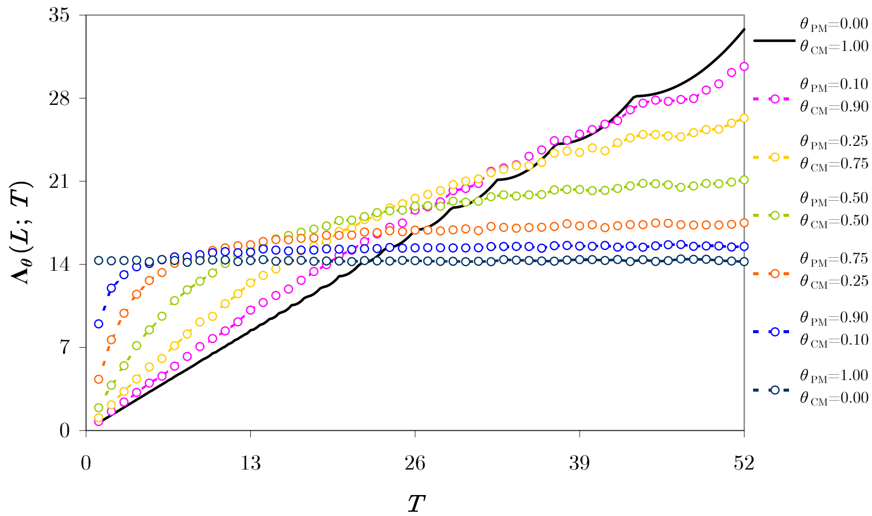

Figure 3.7: Effect of PM and CM age reduction factors on $\Lambda_{\boldsymbol{\theta}}(L;T)$ ($\boldsymbol{\theta} = (2, 20, \theta_{_{\rm PM}}, \theta_{_{\rm CM}})$, $L = 260$ and $M = 1000$).

the *idealized view* is adopted to model PMs and CMs. We note that decreasing the effectiveness of PM together with increasing the effectiveness of CM, makes $\Lambda_{\boldsymbol{\theta}}(L;T)$ independent of T. Such instances favor running the equipment to failure, provided there is no significant difference between the cost of PMs and CMs.

Kijima-I Age-Reduction Model

The results we have presented so far in this section assume the *Kijima-II* virtual age-reduction model. Here, we present simulation results under *Kijima-I*. Varying the degree of PM effectiveness, Figure 3.8 compares $\Lambda_{\boldsymbol{\theta}}(L;T)$ from *Kijima-I* and *Kijima-II*, when CMs are modeled as minimal repairs. Solid lines indicate theoretical values of $\Lambda_{\boldsymbol{\theta}}(L;T)$ under *Kijima-II*, which are obtained using equation (3.16). Circles on dotted lines represent estimated values of $\Lambda_{\boldsymbol{\theta}}(L;T)$ for the *Kijima-I* age-reduction model. We use Algorithm 1 and generate $M = 1000$ independent sample paths to report estimates for the expected number of failures under *Kijima-I*.

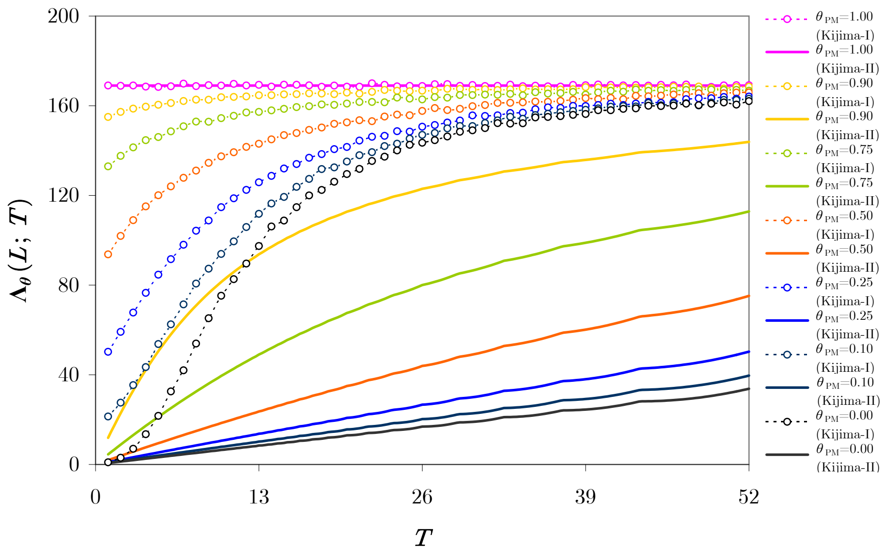

Figure 3.8: Effect of imperfect PMs on $\Lambda_{\boldsymbol{\theta}}(L;T)$ under the Kijima-I and Kijima-II age-reduction models, when CMs are modeled as minimal repairs ($\boldsymbol{\theta} = (2, 20, 0, \theta_{\rm CM})$, $L = 260$ and $M = 1000$ – for *Kijima-I*).

Recall that for the *Kijima-II* model (see equations (2.5) and (2.6)), PMs and CMs remove a portion of the virtual age prior to the maintenance. However, under the *Kijima-I* age-reduction model, PMs and CMs only remove a fraction of the virtual age added since the last maintenance (refer to equations (2.3) and (2.4)). In other words, a PM or CM under *Kijima-I* cannot remove the *damages* accumulated prior to the last maintenance (PM or CM) (Kijima 1989). Therefore, for general values of age reduction factors, the expected number of failures under *Kijima-I* are larger than under *Kijima-II*. The expected number of failures for both the models are equal when (i) $\theta_{\rm CM} = 0$ and $\theta_{\rm PM} = 0$, and (ii) $\theta_{\rm CM} = 1$ and $\theta_{\rm PM} = 1$.

We next study the effect of $\theta_{\rm CM}$ on the expected number of failures. We assume that PMs are perfect and consider the *Kijima-I* age-reduction model. Figure 3.9 plots estimated values of $\Lambda_{\boldsymbol{\theta}}(L;T)$ as a function of T. We observe that the expected number of equipment failures decreases as CMs become more effective.

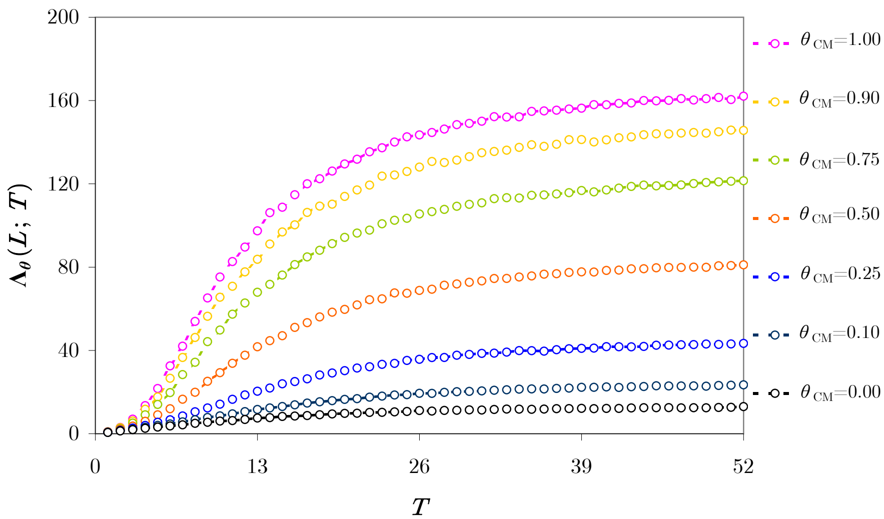

Figure 3.9: Effect of imperfect CMs on $\Lambda_{\boldsymbol{\theta}}(L;T)$ under the *Kijima-I* age-reduction model, when PMs are perfect ($\boldsymbol{\theta} = (2, 20, 0, \theta_{\text{CM}})$, $L = 260$ and $M = 1000$).

Figure 3.10 shows the combined effect of both age reduction factors. It is interesting to note that when CMs become more effective than PMs then running the equipment to failure automatically becomes the optimal maintenance policy. However, if such situations are discovered in practice, system operators should seek to improve the effectiveness of the current PM procedures.

3.3 Computational Results

We assume that $\boldsymbol{\Theta}$ is a random variable and estimate its posterior distribution, $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ through the Gibbs sampling procedure described in Section 2.4. Therefore,

$$\Lambda(L;T) = \mathbb{E}\left[\Lambda_{\boldsymbol{\Theta}}(L;T)\right] = \mathbb{E}\left[\mathbb{E}\left[\mathcal{N}(L;T) \mid \boldsymbol{\Theta}\right]\right], \tag{3.17}$$

and

$$\Lambda_2(L;T) = \mathbb{E}\left[\Lambda_{(2,\boldsymbol{\Theta})}(L;T)\right] = \mathbb{E}\left[\mathbb{E}\left[\mathcal{N}(L;T)^2 \mid \boldsymbol{\Theta}\right]\right]. \tag{3.18}$$

Two expectation operators appear on the right-hand side of equations (3.17) and

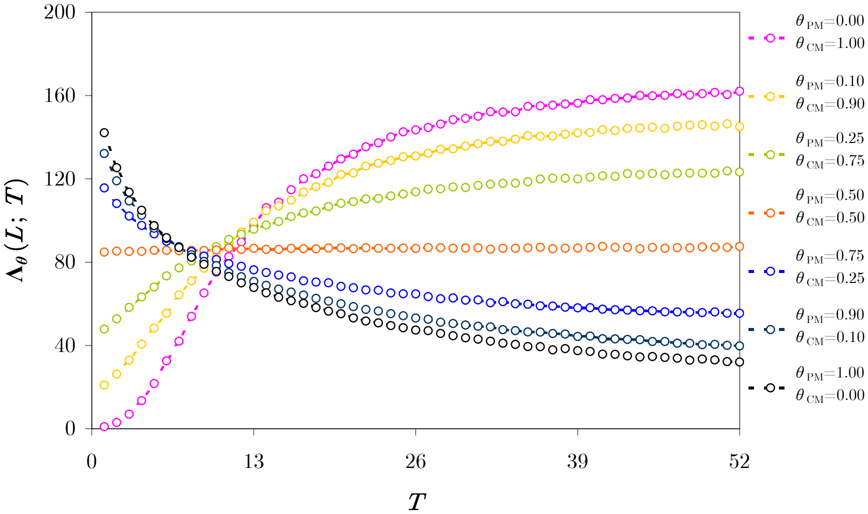

Figure 3.10: Effect of PM and CM age reduction factors on $\Lambda_{\boldsymbol{\theta}}(L;T)$ under the *Kijima-I* virtual age model ($\boldsymbol{\theta} = (2, 20, \theta_{\text{PM}}, \theta_{\text{CM}})$, $L = 260$ and $M = 1000$).

(3.18). The outer expectation is with respect to $\boldsymbol{\Theta}$ and the inner expectation is with respect to $\mathcal{N}(L;T)$ for a fixed value of $\boldsymbol{\Theta} = \boldsymbol{\theta}$. In order to estimate $\Lambda(L;T)$ and $\Lambda_2(L;T)$, we generate N independent samples from the posterior distribution of $\boldsymbol{\Theta}$. Let $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots, \boldsymbol{\theta}^{(N)}$ be the sampled observations from $\pi_{\boldsymbol{\Theta}|\,\mathbf{X}}(\cdot)$. For each sampled $\boldsymbol{\theta}^{(n)}$, $n = 1, 2, \ldots, N$, we use Algorithm 1 to generate M independent sample paths. Estimates for $\Lambda_{\boldsymbol{\theta}^{(n)}}(L;T)$ and $\Lambda_{(2,\boldsymbol{\theta}^{(n)})}(L;T)$ are obtained using equations (3.11) and (3.12), with $\boldsymbol{\theta} = \boldsymbol{\theta}^{(n)}$, $n = 1, 2, \cdots, N$. Equations (3.19) and (3.20) provide our estimators for $\Lambda(L;T)$ and $\Lambda_2(L;T)$, where $\hat{\mathcal{N}}^{(m)}_{\boldsymbol{\theta}^{(n)}}(L,T)$ denotes the number of failures in the m^{th} simulated sample path, for the n^{th} sampled value of $\boldsymbol{\theta}$.

$$\hat{\Lambda}(L;T) = \frac{1}{N\ M} \sum_{n=1}^{N} \sum_{m=1}^{M} \hat{\mathcal{N}}^{(m)}_{\boldsymbol{\theta}^{(n)}}(L;T) \tag{3.19}$$

$$\hat{\Lambda}_2(L;T) = \frac{1}{N\ M} \sum_{n=1}^{N} \sum_{m=1}^{M} \left(\hat{\mathcal{N}}^{(m)}_{\boldsymbol{\theta}^{(n)}}(L;T)\right)^2 \tag{3.20}$$

Algorithm 2 outlines our procedure to generate M independent sample paths for each of the N sampled observation of $\boldsymbol{\theta}$ from $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$, and forms estimates for $\Lambda(L;T)$ and $\Lambda_2(L;T)$.

Algorithm 2 Estimation of $\Lambda(L;T)$ and $\Lambda(L;T)$

Input: $L,\ T,\ \pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot),\ R(\cdot),\ R^{-1}(\cdot),\ M,\ N,\ v_0$
Output: $\hat{\Lambda}(L;T),\ \hat{\Lambda}_2(L;T)$

for $n = 1, 2, \ldots, N$ **do**
  $\boldsymbol{\theta}^{(n)} \leftarrow \pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$
  for $m = 1, 2, \ldots, M$ **do**
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(n)}$; $\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T) \leftarrow 0$; $j \leftarrow 1$; $v \leftarrow v_0$; $t \leftarrow 0$; $t' \leftarrow 0$;
    while $(t < L)$ **do**
      $U \leftarrow U(0,1)$
      $x \leftarrow -v + R^{-1}\left[R(v) - \ln U\right]$
      $t' \leftarrow t + x$
      if $(t' < jT \text{ AND } t' < L)$ **then**
        $\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T) \leftarrow \hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T) + 1$
        $v \leftarrow \theta_{\text{CM}}\,(v + x)$
        $t \leftarrow t'$
      else
        if $(jT \leq L)$ **then**
          $v \leftarrow \theta_{\text{PM}}\,(v + jT - t)$
          $t \leftarrow jT$
          $j \leftarrow j + 1$
        else
          $v \leftarrow (v + L - t)$
          $t \leftarrow L$
        end if
      end if
    end while
  end for
  $\hat{\Lambda}_{\boldsymbol{\theta}^{(n)}}(L;T) \leftarrow \frac{1}{M}\sum_{m=1}^{M} \hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T)$
  $\hat{\Lambda}_{(2,\boldsymbol{\theta}^{(n)})}(L;T) \leftarrow \frac{1}{M}\sum_{m=1}^{M} \left(\hat{\mathcal{N}}_{\boldsymbol{\theta}}^{(m)}(L,T)\right)^2$
end for
$\hat{\Lambda}(L;T) \leftarrow \frac{1}{N}\sum_{n=1}^{N} \hat{\Lambda}_{\boldsymbol{\theta}^{(n)}}(L,T)$
$\hat{\Lambda}_2(L;T) \leftarrow \frac{1}{N}\sum_{n=1}^{N} \hat{\Lambda}_{(2,\boldsymbol{\theta}^{(n)})}(L;T)$

3.3.1 Example 1: Syringe-Driver Infusion Pump Data from Baker (1991)

In this section, we present optimization results for the subset of syringe-driver infusion pumps dataset from Baker (1991), which is studied by Jack (1997) and Jack (1998). Posterior density estimates of $\boldsymbol{\theta}$ for this dataset are discussed in Section 2.5.1. Recall that for this dataset, we initialize the Gibbs sampler at two different values, (a) $\boldsymbol{\theta}^{\mathbf{0}} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta}^{\mathbf{0}} = \boldsymbol{\theta}^{*} = (2.482, 1057.71, 0.789, 1)$, where $\boldsymbol{\theta}^{*}$ is the MLE of $\boldsymbol{\theta}$. As a result, we obtain two estimates of the posterior distribution, $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$, which we denote by $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(b)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$, where the superscript (a) and (b) indicates the two initial values.

In Section 2.5.1, we establish that there is no significant difference between $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(b)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$, suggesting that the Gibbs sampler has converged to the unique limiting distribution independent of the initial value. We next investigate whether $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(b)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ provide consistent estimates for $\Lambda(L;T)$ and $\Lambda_2(L;T)$. We fix $N = 200$ and $M = 500$ in Algorithm 2, and estimate $\Lambda(L;T)$ and $\Lambda_2(L;T)$ under the two posterior density estimates. Figure 3.11 compares $\Lambda(L;T)$ (see (i) and (ii)) and $\Lambda_2(L;T)$ (see (iii) and (iv)) under $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(b)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$, for different values of L. The results once again suggest that the two estimated posterior distributions, $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(b)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ do not differ significantly. In the following subsections, we only consider $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$. For simplicity, we drop the superscript (a) from $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ to emphasize that the posterior distribution is independent of the initial value.

Jack (1998) only reports MLE of $\boldsymbol{\theta}$ and does not present optimization results for the dataset. But we take this opportunity to discuss the effect of the optimization model parameters on the optimal periodic PM policy. In this direction, assume that a PM is not required at the end of maintenance planning horizon L, and consider

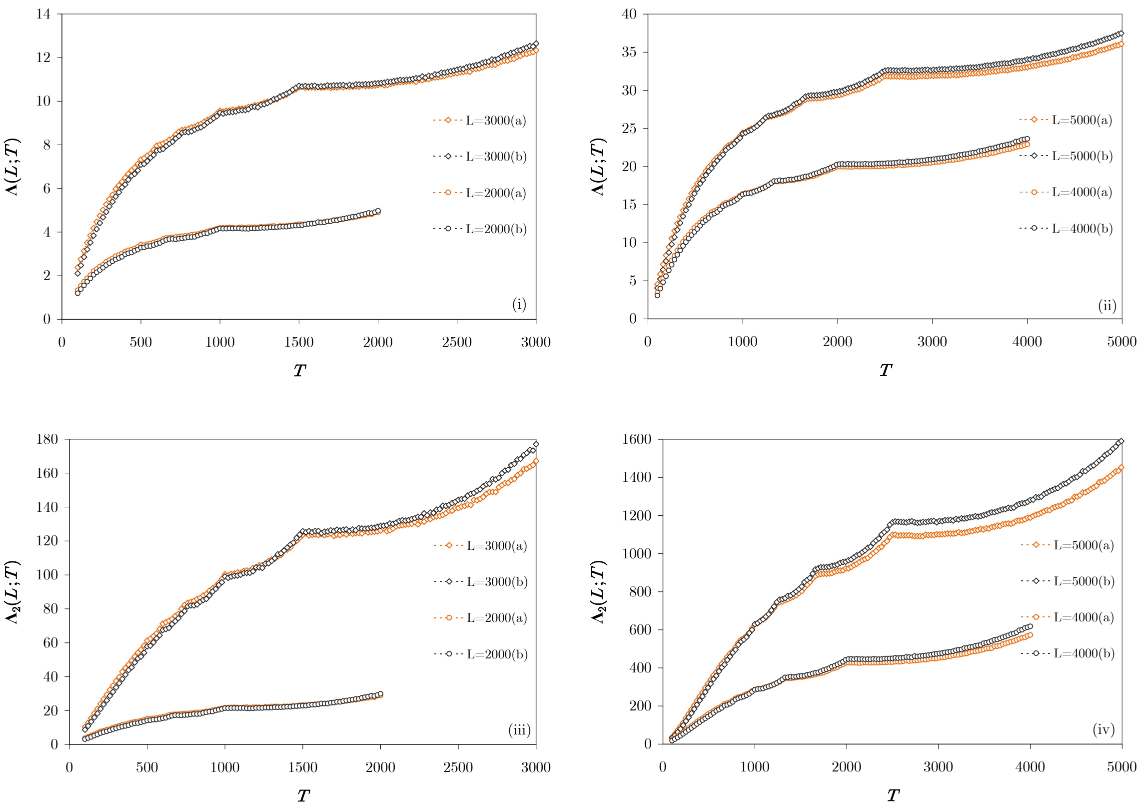

Figure 3.11: Comparison of estimates of $\Lambda(L;T)$ (see (i) and (ii)) and $\Lambda_2(L;T)$ (see (iii) and (iv)) when $N = 200$ independent samples of $\boldsymbol{\theta}$ are drawn from posterior distributions $\pi^{(a)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(b)}_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$. Superscripts (a) and (b) indicate that the Gibbs sampler is initialized at (a) $\boldsymbol{\theta^0} = (2.5, 2500, 0.5, 0.5)$ and (b) $\boldsymbol{\theta^0} = \boldsymbol{\theta^*} = (2.482, 1057.71, 0.789, 1)$, where $\boldsymbol{\theta^*}$ is the MLE of $\boldsymbol{\theta}$.

the following optimization problem from Section 3.1.1:

$$z_3^*(L) = \min_{T\in[0,L]} \left\{ \left(\left\lceil \frac{L}{T} \right\rceil - 1 \right) C_{\mathrm{PM}} + \left(\left(1 + \frac{\alpha}{2} \right) \Lambda(L;T) + \frac{\alpha}{2} \Lambda_2(L;T) \right) C_{\mathrm{CM}} \right\}. \tag{3.21}$$

Let $C_{\mathrm{R}} = C_{\mathrm{CM}}/C_{\mathrm{PM}}$ be the ratio of expected corrective to expected preventive maintenance cost. We note that the optimal solution to (3.21) is indifferent to the maintenance cost units, and depends on C_{R}. Therefore, we study the effect of L, α and C_{R} on the optimal periodic PM policy, obtained by solving

$$\tilde{z}_3^*(L) = \min_{T\in[0,L]} \left\{ \left(\left\lceil \frac{L}{T} \right\rceil - 1 \right) + \left(\left(1 + \frac{\alpha}{2} \right) \Lambda(L;T) + \frac{\alpha}{2} \Lambda_2(L;T) \right) C_{\mathrm{R}} \right\}. \tag{3.22}$$

Effect of Maintenance Planning Horizon, L

In this research, we focus on PM policies for a given finite planning horizon, L. Typically, L is fixed at a strategic level and is primarily governed by the maximum useful life of the equipment under consideration. But it may be important to know the impact of L on the optimal PM interval. Figure 3.12 plots the total maintenance cost as a function of T for different values of L when $C_{\mathrm{R}} = 2$. Since we use simulation to estimate $\Lambda(L;T)$ and $\Lambda_2(L;T)$, the total maintenance cost is reported for a discrete set of values of T over L. Risk neutral costs and optimal PM intervals are shown in Figures 3.12(a) and 3.12(b), respectively. Figures 3.12(c) and 3.12(d) display the risk averse costs and optimal PM intervals for different values of L at $\alpha = 5\%$.

Unless restricted by L, the optimal PM interval either decreases or remains unchanged with increasing L. Intuitively, it makes sense to do PM at frequent intervals if we expect to use the equipment for a longer period of time, and the equipment deteriorates with time and PM is effective.

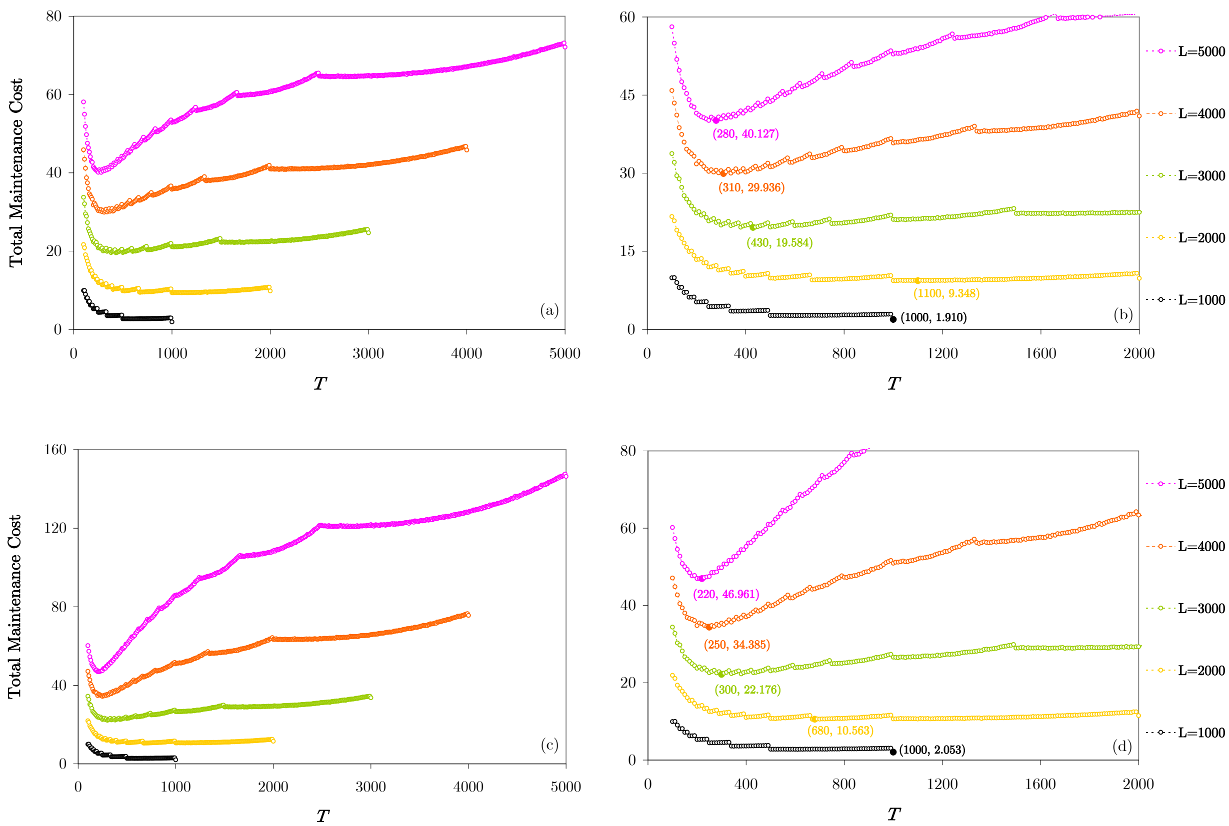

Figure 3.12: Effect of L on the total expected maintenance cost: (a) risk-neutral cost, (b) risk-neutral optimal PM intervals, (c) risk-averse cost at $\alpha = 5\%$, and (d) risk-averse optimal PM intervals at $\alpha = 5\%$.

Effect of Risk Factor, α

Figure 3.13 shows the effect of the risk factor α on the total expected maintenance cost when $L = 2000$ and $C_{\text{R}} = 2$. Recall that the j^{th} equipment failure over L in the *risk-averse* model, costs $\mathcal{C}_{\text{CM}} + \alpha\, j\, \mathcal{C}_{\text{CM}}$ (see Section 3.1.1). Therefore, increasing the value of α results in larger CM costs and hence a larger total maintenance cost over L, which in general, leads to smaller optimal PM intervals.

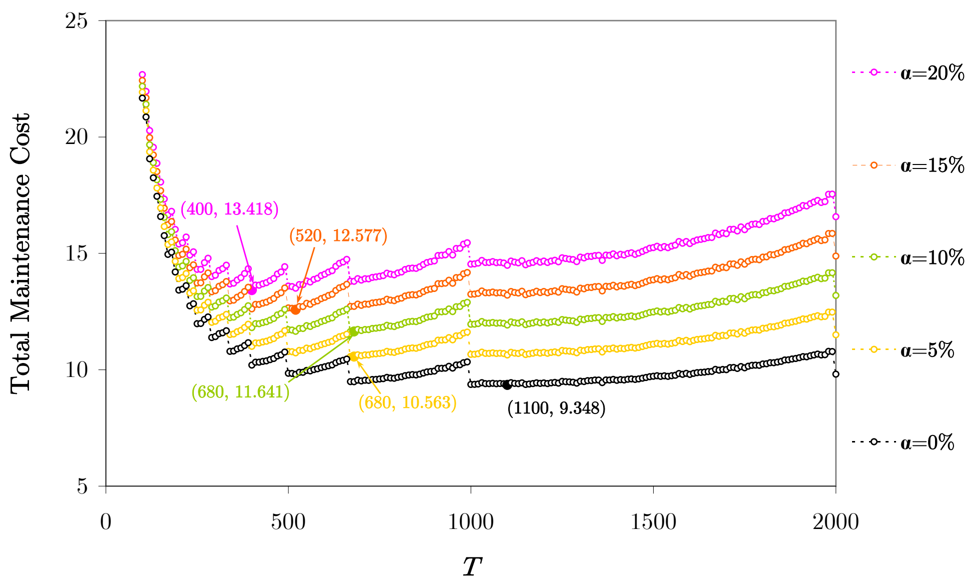

Figure 3.13: Effect of α on the total expected maintenance cost ($L = 2000$ and $C_{\text{R}} = 2$).

Effect of Expected CM to PM cost ratio, C_{R}

Figure 3.14 shows the effect of C_{R} on the total expected maintenance cost with $L = 2000$ at $\alpha = 5\%$. We understand that effective PMs help to reduce the probability of operational equipment failures. If the cost associated with a CM is very high as compared to the cost of a PM, scheduling more frequent PMs become an attractive option. Therefore, increasing C_{R}, in general results in shorter optimal PM intervals.

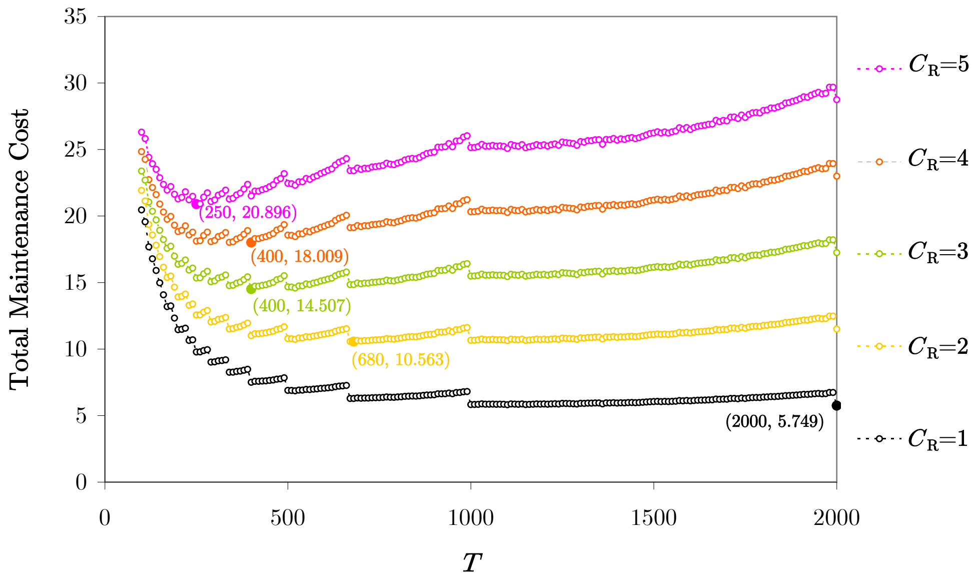

Figure 3.14: Effect of C_{R} on the total expected maintenance cost ($L = 2000$ and $\alpha = 5\%$).

Maximum Likelihood vs. Bayes' Optimal PM Intervals

For a given age-reduction model, the failure rate and maintenance effectiveness parameters, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q, \theta_{\text{PM}}, \theta_{\text{CM}})$, completely define the counting process $\{\mathcal{N}(L;T), L \geq 0\}$. Two key statistics of the counting process, $\Lambda(L;T)$ and $\Lambda_2(L;T)$, together with the parameters of the optimization model (L, α and C_{R}), dictate the optimal PM policy. In practical applications, true value of $\boldsymbol{\theta}$ is unknown and it must be estimated. In this research, we estimate $\boldsymbol{\theta}$ in the Bayesian setting through the Gibbs sampling procedure described in Section 2.4. However, other researchers have estimated the parameters of interest using a maximum likelihood (ML) approach. Therefore, we compare the optimization results when $\boldsymbol{\theta}$ is estimated through ML and Bayesian methods.

Two different types of MLEs of $\boldsymbol{\theta}$ are considered. In the first, both PM and CM can be imperfect, whereas the second adopts an *idealized view* on the two maintenance actions. Recall that $\boldsymbol{\theta^*}$ denotes MLE in the former, and we let $\boldsymbol{\theta}^*_{iv}$ be the MLE in the latter case. For the dataset under discussion, Jack (1998) provides $\boldsymbol{\theta^*}$. We have verified $\boldsymbol{\theta^*} = (2.482, 1057.71, 0.789, 1)$. In addition, we find that the MLE under the *idealized view* is $\boldsymbol{\theta}^*_{iv} = (1.126, 376.92, 0, 1)$. Figure 3.15 compares failure rate function, $r(\cdot)$ given by the two MLEs. We note that $\boldsymbol{\theta}^*_{iv}$ in comparison with $\boldsymbol{\theta^*}$ yields smaller *shape* and *scale* parameters. Since the *shape* parameters exceed 1, both the MLEs imply increasing failure rate. However, the smaller *shape* parameter for $\boldsymbol{\theta}^*_{iv}$ suggests that $r(\cdot)$ increases at a slower rate. On the other hand, the smaller *scale* parameter in $\boldsymbol{\theta}^*_{iv}$ indicates a higher failure rate in the beginning.

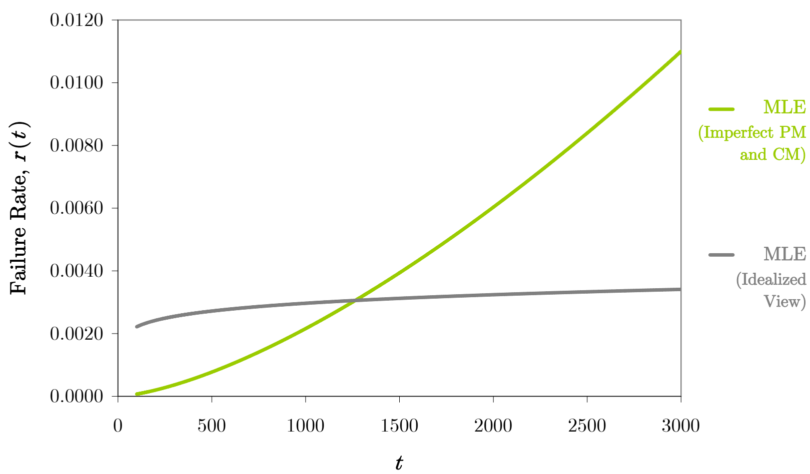

Figure 3.15: Comparison of failure rate, $r(\cdot)$ given by two different types of MLEs.

As mentioned earlier, closed-form expressions for $\Lambda(L;T)$ and $\Lambda_2(L;T)$ exist when CMs are minimal repairs, which is an explicit assumption under the *idealized view*. Since $\boldsymbol{\theta}^*$ also implies minimal CMs ($\theta^*_{\mathrm{CM}} = 1$), we analytically compute $\Lambda(L;T)$ and $\Lambda_2(L;T)$ for both the MLEs. In the Bayesian context, estimates for $\Lambda(L;T)$ and $\Lambda_2(L;T)$ are obtained through Algorithm 2 with $N = 200$ and $M = 500$. Figure 3.16 compares $\Lambda(L;T)$ and $\Lambda_2(L;T)$ for $L = 3000$ under the ML and Bayesian settings.

Once again solid lines indicate theoretical values and circles on the dotted line represents simulated values.

In general, estimates of $\Lambda(L;T)$ and $\Lambda_2(L;T)$ from $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ can be arbitrarily different from the corresponding values under $\boldsymbol{\theta}^*$. However, in this example, we note that the estimated and the theoretical values are consistent with each other for $T \leq 1500$. For $T > 1500$, we notice that the difference between the two values increases with increasing T. In addition, we expect this difference to grow with increasing L. Furthermore, in comparison with $\boldsymbol{\theta}^*$ and samples of $\boldsymbol{\theta}$ from $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$, MLE under the *idealized view*, $\boldsymbol{\theta}^*_{iv}$, initially yields larger values of $\Lambda(L;T)$ and $\Lambda_2(L;T)$, followed by smaller values. This is due to the combined effect of the failure rate characteristics (discussed earlier) and the perfect PM assumption under $\boldsymbol{\theta}^*_{iv}$.

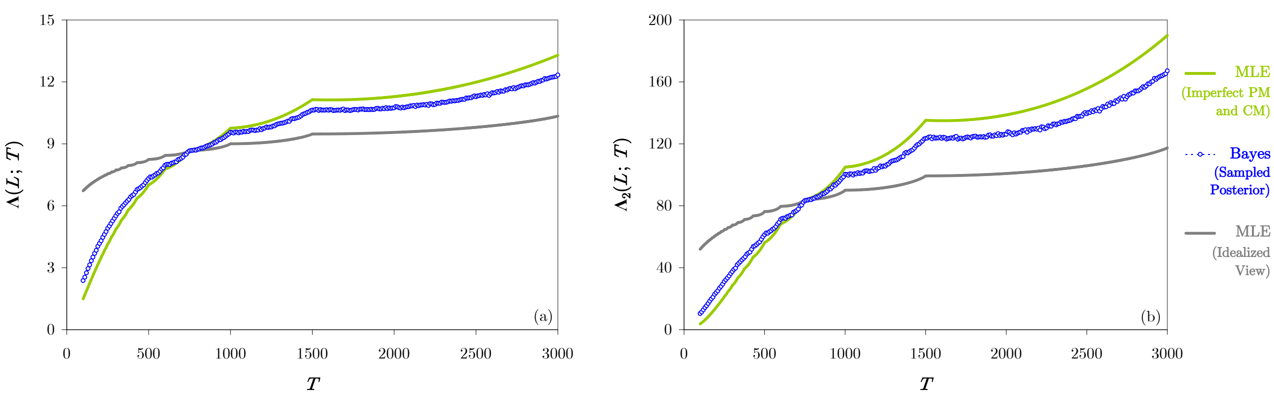

Figure 3.16: Comparison of (a) $\Lambda(L;T)$ and (b) $\Lambda_2(L;T)$ for maximum likelihood and Bayesian estimates of $\boldsymbol{\theta}$.

Figure 3.17 compares the expected total maintenance cost under ML and Bayesian estimates of $\boldsymbol{\theta}$. Simulated and theoretical values of the expected total cost are reported for $L = 3000$, $C_{\mathrm{R}} = 2$ and $\alpha = 5\%$. For the given values of the optimization model parameters, samples of $\boldsymbol{\theta}$ from $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\boldsymbol{\theta}^*$ result in equal optimal PM intervals. However, the corresponding optimal expected total maintenance costs differ. In comparison, the optimal solution, $\boldsymbol{\theta}^*_{iv}$, under the *idealized view* suggests a longer PM interval at a higher cost. Interestingly, when $T = 750$, the periodic PM policy is nearly indifferent under ML and Bayesian settings.

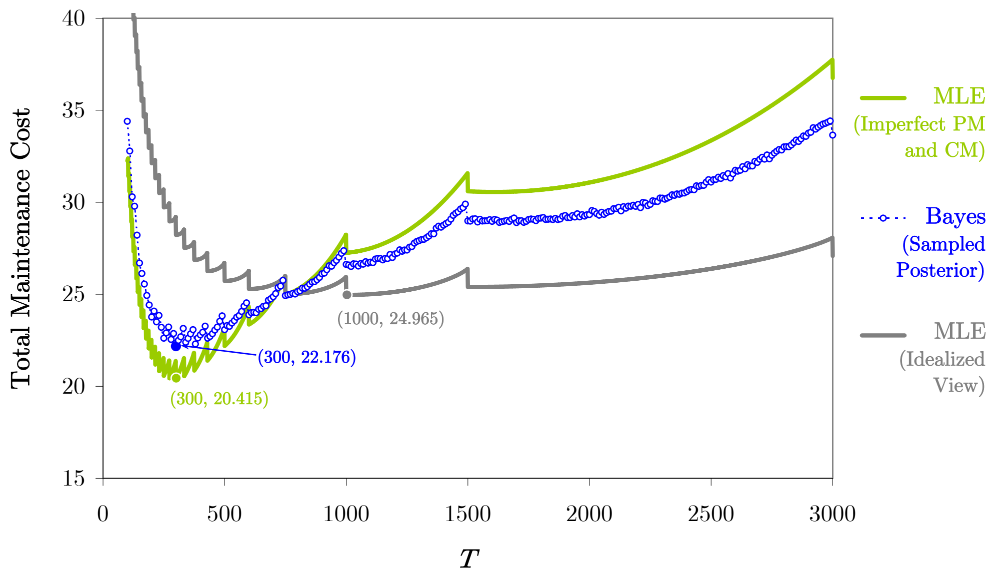

Figure 3.17: Total expected maintenance cost and the optimal periodic PM policy under maximum likelihood and Bayesian estimates of $\boldsymbol{\theta}$ ($L = 3000$, $C_{\mathrm{R}} = 2$ and $\alpha = 5\%$).

3.3.2 Example 2: Simulated Maintenance Dataset from Yu et al. (2008)

We next discuss the optimization results for the simulated dataset in Yu et al. (2008). Posterior density estimates for the dataset are discussed in Section 2.5.2. Recall that this dataset considers the *Kijima-I* age-reduction model and the associated true parameter, $\boldsymbol{\theta}^{\text{true}} = \left(\theta_1^{\text{true}}, \theta_2^{\text{true}}, \theta_{\text{PM}}^{\text{true}}, \theta_{\text{CM}}^{\text{true}}\right) = (2.2, 1, 0.8, 0.3)$. We note that $\theta_{\text{CM}}^{\text{true}} < \theta_{\text{PM}}^{\text{true}}$, and therefore CMs are more effective than PMs. Our simulation experiences from Section 3.2.1 show that in such situations, it is not economical to schedule a PM. In other words, if CMs are more effective than PMs then under *Kijima-I* age-reduction model, running the equipment to failure automatically becomes an optimal maintenance policy. Perhaps due to this reason, Yu et al. (2008) do not present optimization results for their dataset. Although we expect a *trivial* optimal PM policy under $\boldsymbol{\theta}^{\text{true}}$, this dataset provides an opportunity to check if optimal PM policies under ML and Bayesian settings agree with the *true* optimal PM policy.

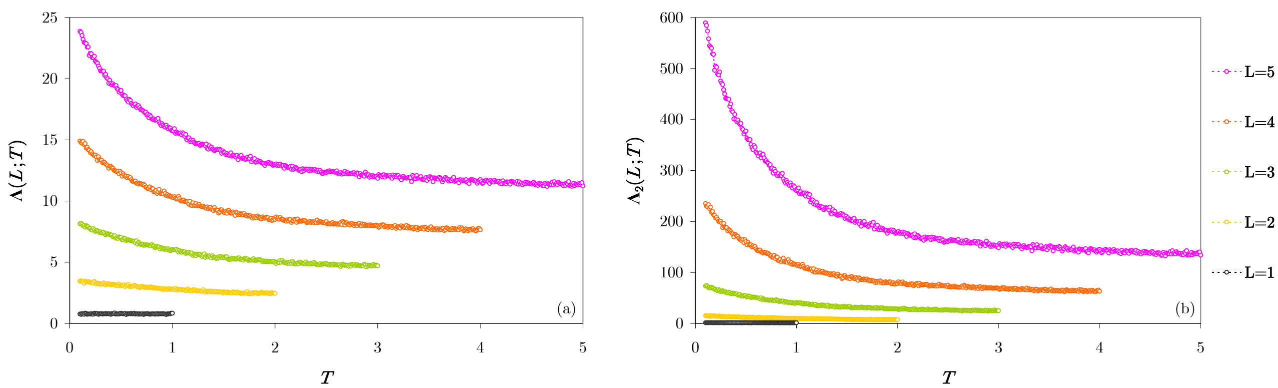

Figure 3.18: Estimated values of (a) $\Lambda(L;T)$ and (b) $\Lambda_2(L;T)$ under $\boldsymbol{\theta}^{\text{true}} = (2.2, 1, 0.8, 0.3)$.

Closed-form expressions for $\Lambda(L;T)$ and $\Lambda_2(L;T)$ under the *Kijima-I* model do not exist for $\boldsymbol{\theta}^{\text{true}}$. Therefore, we use Algorithm 1 and generate $M = 1000$ independent sample paths to obtain corresponding estimates under $\boldsymbol{\theta}^{\text{true}}$. Figure 3.18 shows the estimated values of $\Lambda(L;T)$ and $\Lambda_2(L;T)$ for different L. Recall that in the *Kijima-I* age-reduction model, a given PM or CM can only remove a fraction of the virtual age added since the last maintenance. Consequently, if PMs are ineffective (as compared to CMs) and performed more often, *damages* accumulates quickly, resulting in frequent failures. Therefore, for the dataset under consideration, both $\Lambda(L;T)$ and $\Lambda_2(L;T)$ decrease with increasing T over L.

We next turn to the optimization results. In this direction, we first obtain two different types of MLEs of $\boldsymbol{\theta}$. When CMs and PMs are assumed imperfect, the

MLE of $\boldsymbol{\theta}$, $\boldsymbol{\theta}^{*} = (2.616, 1.096, 1, 0)$, and MLE of $\boldsymbol{\theta}$ under the *idealized view* yields $\boldsymbol{\theta}^{*}_{iv} = (1.828, 0.633, 0, 1)$. We note that the effectiveness of the two maintenance actions in $\boldsymbol{\theta}^{*}$ is just the opposite of the assumed effectiveness under the *idealized view*.

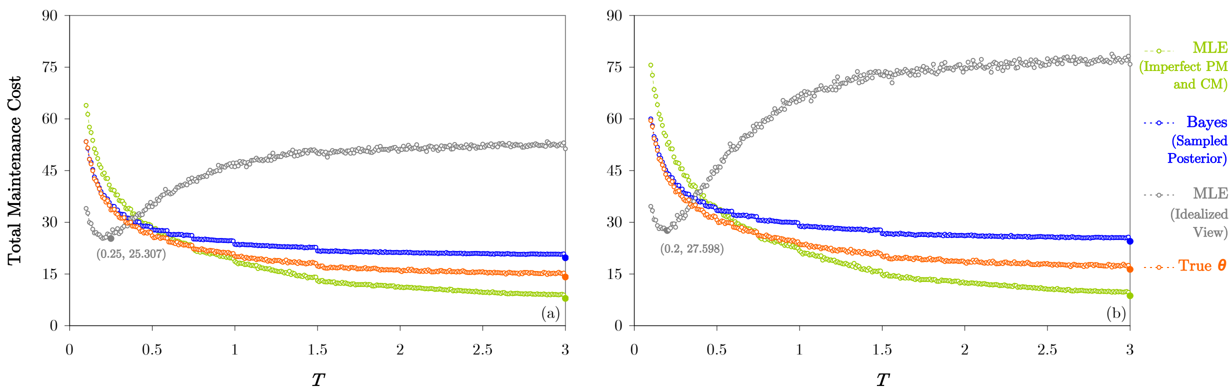

Figure 3.19: Total expected maintenance cost and the optimal periodic PM policy under $\boldsymbol{\theta}^{\text{true}}$, sampled posterior and the two MLEs ($L = 3$, $C_{\text{R}} = 3$, and (a) $\alpha = 0\%$ (b) $\alpha = 5\%$)

Figure 3.19 compares the expected total maintenance cost and the optimal PM policy under $\boldsymbol{\theta}^{\text{true}}$, sampled posterior and the two MLEs under both the (a) risk-neutral and (b) risk-averse formulations. The expected total maintenance costs are reported for $L = 3$, $C_{\text{R}} = 3$, and (a) $\alpha = 0\%$ and (b) $\alpha = 5\%$. We set $N = 200$ and $M = 500$ in Algorithm 2, and approximate $\Lambda(L;T)$ and $\Lambda_2(L;T)$ under posterior density estimates, $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$. Estimated values of $\Lambda(L;T)$ and $\Lambda_2(L;T)$ for the two MLEs are obtained through Algorithm 1 with $M = 1000$. The estimates of $\Lambda(L;T)$ and $\Lambda_2(L;T)$ together with optimization model parameters provide the expected total maintenance cost. We note that samples of $\boldsymbol{\theta}$ from $\pi_{\boldsymbol{\Theta}|\mathbf{X}}(\cdot)$ and $\boldsymbol{\theta}^{*}$ correctly identify the *true* PM policy. However, $\boldsymbol{\theta}^{*}_{iv}$ fails to discover the *true* PM policy.

3.3.3 Example 3: STP Dataset

In this section, we discuss the PM optimization results for the water transfer pumps dataset from STP, posterior density estimates of which are presented in Section 2.5.3. In general, for all STP datasets, we assume that a PM is required at the end of the maintenance planning horizon L. Therefore, we consider the following optimization problem from Section 3.1.1:

$$z_3^*(L) = \min_{T \in [0,L]} \left\{ \left\lceil \frac{L}{T} \right\rceil C_{\text{PM}} + \left(\left(1 + \frac{\alpha}{2}\right) \Lambda(L;T) + \frac{\alpha}{2} \Lambda_2(L;T) \right) C_{\text{CM}} \right\}. \tag{3.23}$$

The average cost of a PM (C_{PM}) and a CM (C_{CM}), as tracked in STP's work management system are \$581 and \$2038, respectively. In addition, the present worth of the total cost of maintenance for a pump, over the last 20 years is approximately \$28,000. As mentioned earlier, the PMs are performed at an interval of 104 weeks (2 years).

STP maintains both probabilistic risk assessment (PRA) and balance of plant (BOP) risk models. The former helps to estimate on-line plant safety and the latter is used to assess production performance. The water transfer pumps under consideration bear a *not risk significant* (NRS) rank. In general, there is no safety and production risk associated with an operational failure of NRS equipment. We verify that no PRA/BOP event is linked to the pumps. Therefore, a small value of the risk factor α, say 5%, seems appropriate.

Recall that in Section 2.5.3, we use different sets of priors for the two age reduction factors. We show that a uniform prior on CM results in a right-skewed posterior, indicating that the dataset supports effective CMs. However, the dataset does not provide strong evidence for effective PM. In addition, uniform and skew priors on the two age reduction factors did not have a significant effect on the posterior densities of the two failure rate parameters. Let $\pi^{(u)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(s)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$ be the posterior estimates of $\mathbf{\Theta}$ for uniform and skew priors, respectively. In the following, we compare estimated values of $\Lambda(L;T)$ and $\Lambda_2(L;T)$ under $\pi^{(u)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(s)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$. We fix $N = 400$ and $M = 500$ in Algorithm 2 and approximate $\Lambda(L;T)$

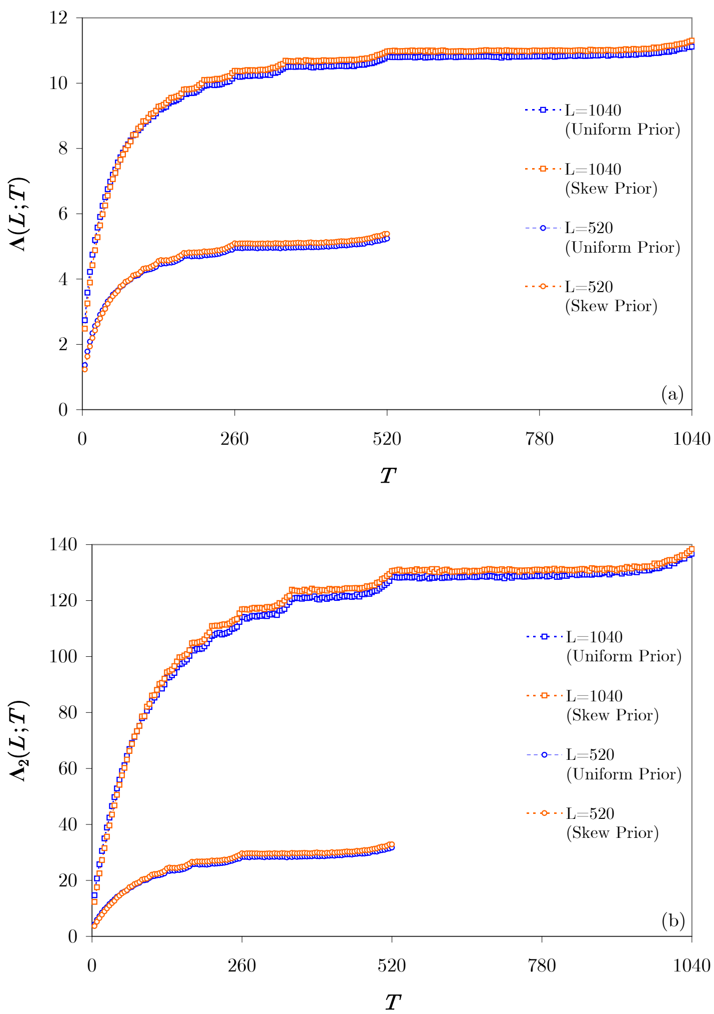

Figure 3.20: Comparison of (a) $\Lambda(L;T)$ and (b) $\Lambda_2(L;T)$ estimates when $N = 400$ independent samples of $\boldsymbol{\theta}$ are drawn from posterior distributions $\pi^{(u)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(s)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$. Superscripts (u) and (s) indicate uniform and skew priors for the two age reduction factors, respectively.

and $\Lambda_2(L;T)$ for the two sets of posteriors. Figure 3.20 displays estimated values of

$\Lambda(L;T)$ and $\Lambda_2(L;T)$ under $\pi^{(u)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$ and $\pi^{(s)}_{\mathbf{\Theta}|\mathbf{X}}(\cdot)$. The results suggest that the two posterior distributions do not differ significantly.

When CMs and PMs are assumed imperfect, we have that the MLE of $\boldsymbol{\theta}$ is $\boldsymbol{\theta^*} = (1.585, 100.21, 0.1025, 0)$. We note that $\boldsymbol{\theta^*}$ suggests perfect CMs in contrast with minimal CM, assumed under the *idealized view*. In addition, $\boldsymbol{\theta^*}$ indicates that PMs are effective but not perfect. We find that the MLE of $\boldsymbol{\theta}$ under the *idealized view*, $\boldsymbol{\theta}^*_{iv} = (1.312, 111.32, 0, 1)$. Figure 3.21 displays *risk-neutral* and *risk-averse* expected total maintenance costs under ML and Bayesian estimates of $\boldsymbol{\theta}$. The costs are reported for $L = 1040$ (weeks), which is the remaining life of the NPPs operated by STP. Table 3.1 provides a summary of the optimal PM policies. The total cost for the current PM interval and the optimal cost are also included.

Interestingly, the current PM interval is justified under each of the ML and Bayesian estimates considered. The optimal PM interval under the *idealized view* happens to be the current PM interval. There is a small difference between the optimal cost and the total cost for the current PM interval, suggesting that the current PM interval under the remaining estimates of $\boldsymbol{\theta}$ is also reasonable, from the perspective of PM optimization.

Estimate	*Risk-Neutral*			*Risk-Averse* ($\alpha = 5\%$)		
	Optimal PM Interval, T^*	Total Cost (\$k) at T^*	Total Cost (\$k) at $T = 104$	Optimal PM Interval, T^*	Total Cost (\$k) at T^*	Total Cost (\$k) at $T = 104$
MLE, $\boldsymbol{\theta}^*$	667.33	22.616	23.667	73.67	28.460	28.579
MLE, $\boldsymbol{\theta}^*_{iv}$	104	23.868	23.868	82.33	28.631	29.063
Bayes, $\pi^{(u)}_{\mathbf{\Theta}\vert\mathbf{X}}(\cdot)$	268.67	22.512	23.072	95.33	27.794	27.810
Bayes, $\pi^{(s)}_{\mathbf{\Theta}\vert\mathbf{X}}(\cdot)$	273	22.860	23.231	82.33	27.961	28.151

Table 3.1: Comparison of optimization results for STP dataset under maximum likelihood and Bayesian estimates.

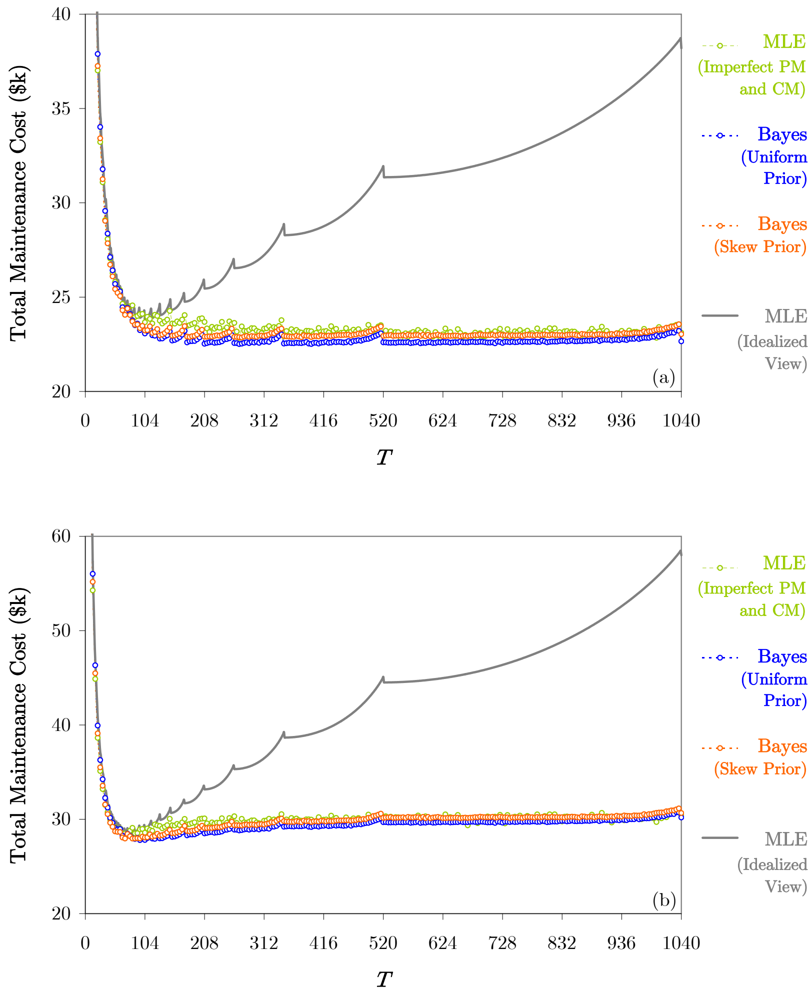

Figure 3.21: Total expected maintenance cost (a) *risk neutral* ($\alpha = 0\%$) (b) *risk averse* ($\alpha = 5\%$) under maximum likelihood and Bayesian estimates ($L = 1040$, $C_{\text{PM}} = \$581$, $C_{\text{CM}} = \$2038$).

Chapter 4

Periodic Preventive Maintenance Optimization Models

In Chapter 3, we introduce our first class of periodic PM optimization models, $\mathcal{M}(1)$, and develop a simulation-based approach to solve the periodic PM optimization problem, based on this class of models. We recall that the need to simulate an equipment failure process arises when closed-form expression for the expected number of failures is not available. We also recall that when the CMs are models as minimal repairs then the number of equipment failures within each PM interval has a Poisson distribution, under both *Kijima-I* and *Kijima-II* virtual age-reduction models. But it is only possible to construct closed-form expressions for the expected number of failures, under the *Kijima-II* virtual age-reduction model. Therefore, the minimal repair assumption under the *Kijima-II* virtual age-reduction model allows us to model and solve the periodic PM optimization problem analytically. Hence, throughout this chapter we assume the *Kijima-II* age-reduction model.

We note that the minimal CM assumption is reasonable in practice for a complex piece of equipment with many parts, where failure of any part may result in equipment failure. Therefore, the PM optimization models presented in this chapter apply to such types of complex equipment. In this chapter, we first introduce our second class of periodic PM optimization models which have two decision variables, the first for the number of PMs and the second for the length of PM intervals, once the number of PMs are fixed. We recall that our first class of periodic

PM optimization models have only one decision variable for the length of the PM intervals, which also determines the number of PMs. Therefore, our second class of periodic PM models are more general than the first class of models. We begin with our second class of periodic PM optimization models and later establish the relationship between the two classes of models.

4.1 Two-Stage Periodic PM Optimization Model

In general, the periodic PM optimization problem can be formulated as a two-stage optimization model. In the first stage, we fix the number of PMs, $n \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers. The second stage then makes decision with respect to T such that $nT \leq L$ and minimizes the expected total cost over $[0, L]$. We require that if $n = 0$ then $T = L$ be the only feasible solution to the two-stage periodic PM model. Recall that in our first class of periodic PM optimization models, $\mathcal{M}(1)$, defined in Section 3.1, we use random variable $\mathcal{N}(L;T)$ to denote the number of equipment failures over the time interval $[0, L]$. We extend the notation from $\mathcal{N}(L;T)$ to $\mathcal{N}(L;n,T)$, to indicate dependence on both n and T. The general two-stage periodic PM policy aims at minimizing the expected value of a random total cost function $\mathcal{C}(L;n,T)$. We write $\mathcal{C}(L;n,T)$ as the sum of two random cost functions, $\mathcal{C}_p(L;n)$ and $\mathcal{C}_c(L;n,T)$. The former denotes the random cost of n PMs and the latter defines the random CM cost of $\mathcal{N}(L;n,T)$ failures on $[0, L]$, for a general periodic PM policy identified by n and T. Since the number of PMs are fixed in the first stage, we note that $\mathcal{C}_p(\cdot)$ is independent of T. Mathematically,

$$z^*(L) = \min_{n \in \mathbb{N}_0} \left\{ \begin{array}{c} \min\limits_{T \in [0,L]} \mathbb{E}\left[\mathcal{C}\left(L;n,T\right)\right] \\ \text{s.t. } nT \leq L \end{array} \right\} \qquad \mathcal{M}(2)$$

$$
\begin{aligned}
&= \min_{n\in\mathbb{N}_0} \left\{ \min_{T\in\left[0,\frac{L}{n}\right]} \mathbb{E}\left[\mathcal{C}\left(L;n,T\right)\right] \right\} \\
&= \min_{n\in\mathbb{N}_0} \left\{ \min_{T\in\left[0,\frac{L}{n}\right]} \mathbb{E}\left[\mathcal{C}_p(L;n) + \mathcal{C}_c(L;n,T)\right] \right\} \\
&= \min_{n\in\mathbb{N}_0} \left\{ \mathbb{E}\left[\mathcal{C}_p(L;n)\right] + \min_{T\in\left[0,\frac{L}{n}\right]} \mathbb{E}\left[\mathcal{C}_c(L;n,T)\right] \right\}, \qquad (4.1)
\end{aligned}
$$

where, we use $\mathcal{M}(2)$ to denote our second class of periodic PM optimization models.

4.1.1 Risk-Averse Formulation

In Section 3.1.1, we introduce a risk-averse formulation from our first class of periodic PM optimization models, $\mathcal{M}(1)$. Here, we present a general two-stage risk-averse periodic PM optimization model under $\mathcal{M}(2)$. As in Section 3.1.1, we let random variables $\mathcal{C}_{\text{PM}}$ and $\mathcal{C}_{\text{CM}}$ denote the cost of performing a PM and a CM, respectively. Let C_{PM} and C_{CM} be the expected cost of a PM and a CM such that, $C_{\text{PM}} = \mathbb{E}\left[\mathcal{C}_{\text{PM}}\right] < \infty$, and $C_{\text{CM}} = \mathbb{E}\left[\mathcal{C}_{\text{CM}}\right] < \infty$. The total expected cost of n PMs, which are fixed at the first stage, is given by

$$
\mathbb{E}\left[\mathcal{C}_p(L;n)\right] = \mathbb{E}\left[n\mathcal{C}_{\text{PM}}\right] = nC_{\text{PM}}. \qquad (4.2)
$$

We assume that $\mathcal{C}_{\text{CM}}$ is independent of the random number of failures, $\mathcal{N}(L;n,T)$. Furthermore, we assume that for any values n and T, we have

$$
\Lambda(L;n,T) = \mathbb{E}\left[\mathcal{N}(L;n,T)\right] < \infty,
$$

and

$$
\Lambda_2(L;n,T) = \mathbb{E}\left[\mathcal{N}(L;n,T)^2\right] < \infty.
$$

We adhere to a notion of *global* risk and adopt the definition of the CM cost function,

$\mathcal{C}_{\text{CM}}(\cdot)$, from equation (3.8). Hence,

$$\begin{aligned}\mathbb{E}\left[\mathcal{C}_{\text{c}}(L;n,T)\right] &= \mathbb{E}\left[\mathcal{C}_{\text{CM}}\left(\mathcal{N}\left(L;n,T\right)\right)\right] \\ &= \mathbb{E}\left[\left(\left(1+\frac{\alpha}{2}\right)\mathcal{N}(L;n,T)+\frac{\alpha}{2}\mathcal{N}(L;n,T)^2\right)\mathcal{C}_{\text{CM}}\right] \\ &= \left(\left(1+\frac{\alpha}{2}\right)\Lambda(L;n,T)+\frac{\alpha}{2}\Lambda_2(L;n,T)\right)C_{\text{CM}}. \end{aligned} \tag{4.3}$$

From equations (4.1), (4.2) and (4.3), we obtain a risk-averse formulation of the two-stage periodic PM optimization problem, which is provided below:

$$z^*(L) = \min_{n\in\mathbb{N}_0}\left\{nC_{\text{PM}} + \min_{T\in\left[0,\frac{L}{n}\right]}\left\{\left(\left(1+\frac{\alpha}{2}\right)\Lambda(L;n,T)+\frac{\alpha}{2}\Lambda_2(L;n,T)\right)C_{\text{CM}}\right\}\right\}. \tag{4.4}$$

We understand that closed-form expressions for the two key parameters, $\Lambda(L;n,T)$ and $\Lambda_2(L;n,T)$, are only available when CMs are modeled as minimal repairs. For general values of CM age reduction factors, $\Lambda(L;n,T)$ and $\Lambda_2(L;n,T)$ must be estimated through simulation. Finally, we note that setting $\alpha = 0$ in (4.4) results in a *risk-neutral* formulation of $\mathcal{M}(2)$.

4.1.2 Imperfect PMs and Minimal CMs

As mentioned earlier, the minimal CM assumption is valid for a complex piece of equipment, with many parts, where failure of any part may result in equipment failure. In addition, operational failures of this type of equipment are addressed by replacing or repairing a few parts, which do not affect the failure characteristics of the equipment under consideration. Furthermore, if the PM in question, does not result in complete overhaul or replacement of the equipment, then it is appropriate to assume that the PM is imperfect. Under these assumptions, we associate an age reduction factor θ^j_{PM}, with the j^{th} realization of the PM, such that $\theta^j_{\text{PM}} \in [0,1]$, $j =$

$1, 2, \ldots, n$, and define

$$\breve{\theta}^{(n)}_{\text{PM}} = \begin{cases} \displaystyle\sum_{j=1}^{n} \prod_{k=j}^{n} \theta^{k}_{\text{PM}} & n \in \mathbb{N} \\ 0 \quad \text{otherwise} \end{cases}. \tag{4.5}$$

Note that for a $n \in \mathbb{N}$,

$$\theta^{n}_{\text{PM}} \left(\breve{\theta}^{(n-1)}_{\text{PM}} + 1 \right) = \breve{\theta}^{(n)}_{\text{PM}}, \tag{4.6}$$

and since $\theta^{n}_{\text{PM}} \in [0, 1]$, we have

$$\left(\breve{\theta}^{(n-1)}_{\text{PM}} + 1 \right) \geq \breve{\theta}^{(n)}_{\text{PM}}. \tag{4.7}$$

If $v(t)$ denotes the virtual age of the equipment at time t and $v(0) = 0$, then the virtual age of the equipment following the j^{th} PM can be derived as follows:

$$\begin{aligned}
v(T) &= \theta^{1}_{\text{PM}} \Big(v(0) + T \Big) = \theta^{1}_{\text{PM}} T = \breve{\theta}^{(1)}_{\text{PM}} T \\
v(2T) &= \theta^{2}_{\text{PM}} \Big(v(T) + T \Big) = \theta^{2}_{\text{PM}} \left(\breve{\theta}^{(1)}_{\text{PM}} + 1 \right) T = \breve{\theta}^{(2)}_{\text{PM}} T \\
&\vdots \\
v(jT) &= \theta^{j}_{\text{PM}} \Big(v((j-1)T) + T \Big) = \theta^{j}_{\text{PM}} \left(\breve{\theta}^{(j-1)}_{\text{PM}} + 1 \right) T \\
&= \breve{\theta}^{(j)}_{\text{PM}} T \quad j = 1, 2, \ldots, n.
\end{aligned}$$

Therefore, for a fixed n and general values of PM age reduction factors, $\theta^{j}_{\text{PM}} \in [0, 1]$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\Lambda(L; n, T) &= \mathbb{E}\left[\mathcal{N}(L; n, T) \right] \\
&= \sum_{j=1}^{n} \Big\{ R\Big(v\left((j-1)T\right) + T \Big) - R\Big(v\left((j-1)T\right) \Big) \Big\} \\
&\qquad + R\Big(v(nT) + L - nT \Big) - R\Big(v(nT) \Big)
\end{aligned}$$

$$= \sum_{j=1}^{n} \left\{ R\left(\left(\breve{\theta}_{\text{PM}}^{(j-1)} + 1 \right) T \right) - R\left(\breve{\theta}_{\text{PM}}^{(j-1)} T \right) \right\}$$
$$+ R\left(\breve{\theta}_{\text{PM}}^{(n)} T + L - nT \right) - R\left(\breve{\theta}_{\text{PM}}^{(n)} T \right), \tag{4.8}$$

where, $R(t) = \displaystyle\int_0^t r(u)\mathrm{d}u$. We note that for a fixed $n \in \mathbb{N}$, there are n PM intervals of length T, followed by a possible interval of length $L - nT \geq 0$, which need not to be less than T. For instance, if $T < \frac{L}{n+1}$ then $L - nT > T$. Furthermore, the second-moment of the number of equipment failures, $\Lambda_2(L; n, T)$, under the minimal CM assumption, satisfies

$$\Lambda_2(L; n, T) = \Lambda(L; n, T) \left(\Lambda(L; n, T) + 1 \right). \tag{4.9}$$

For notational convenience, we write

$$\bar{\Lambda}_\alpha(L; n, T) = \left(1 + \frac{\alpha}{2} \right) \Lambda(L; n, T) + \frac{\alpha}{2} \Lambda_2(L; n, T), \tag{4.10}$$

$$\begin{aligned} \bar{C}_\alpha(L; n, T) &= nC_{\text{PM}} + \left(\left(1 + \frac{\alpha}{2} \right) \Lambda(L; n, T) + \frac{\alpha}{2} \Lambda_2(L; n, T) \right) C_{\text{CM}} \\ &= nC_{\text{PM}} + \bar{\Lambda}_\alpha(L; n, T) C_{\text{CM}}, \end{aligned} \tag{4.11}$$

and represent the risk-averse formulation of the two-stage periodic PM optimization problem from equation (4.4) as,

$$z^*(L) = \min_{n \in \mathbb{N}_0} \left\{ \min_{T \in \left[0, \frac{L}{n}\right]} \bar{C}_\alpha(L; n, T) \right\}. \tag{4.12}$$

From equations (4.9), (4.10) and (4.11), we note that for a fixed value of n, $\bar{C}_\alpha(L; n, T)$ primarily depends on $\Lambda(L; n, T)$. Therefore, we focus on properties of $\Lambda(L; n, T)$. In this direction, the following proposition provides a sufficient condition for the convexity of $\Lambda(L;n,T)$ in T, for a fixed n, and for given values of PM age reduction factors, $\theta^{j}_{\text{PM}} \in [0,1]\,,\ j = 1,2,\ldots,n$.

Proposition 4.1.2.1 *If $r(\cdot)$ is increasing, continuously differentiable, and*

$$r'\left((\breve{\theta}^{(j-1)}_{PM}+1)T\right) \geq \left(\theta^{j}_{PM}\right)^2\ r'\left(\breve{\theta}^{(j)}_{PM}T\right) \geq 0,\ j=1,2,\ldots,n, \tag{4.13}$$

then for a fixed $n \in \mathbb{N}$, $\Lambda\left(L;n,T\right)$ is convex in T for $T \in \left[0,\frac{L}{n}\right]$.

Proof: For a fixed value of $n \in \mathbb{N}$, $\Lambda\left(L;n,T\right)$ is defined for $T \in \left[0,\frac{L}{n}\right]$. Now, $\Lambda\left(L;n,T\right)$ is convex in T, if

$$\frac{\partial^2\Lambda\left(L;n,T\right)}{\partial T^2} \geq 0.$$

Using (4.8) it suffices to show that

$$\begin{aligned}\sum_{j=1}^{n}\Bigg\{ &\left(\breve{\theta}^{(j-1)}_{\text{PM}}+1\right)^2\ r'\left((\breve{\theta}^{(j-1)}_{\text{PM}}+1)T\right) - \left(\breve{\theta}^{(j-1)}_{\text{PM}}\right)^2\ r'\left(\breve{\theta}^{(j-1)}_{\text{PM}}T\right)\Bigg\}\\ &+\left(\breve{\theta}^{(n)}_{\text{PM}}-n\right)^2\ r'\left(\breve{\theta}^{(n)}_{\text{PM}}T+L-nT\right) - \left(\breve{\theta}^{(n)}_{\text{PM}}\right)^2\ r'\left(\breve{\theta}^{(n)}_{\text{PM}}T\right) \geq 0.\end{aligned}$$

Because $\breve{\theta}^{(0)}_{\text{PM}} = 0$, this is equivalent to:

$$\begin{aligned}\sum_{j=1}^{n}\Bigg\{ &\left(\breve{\theta}^{(j-1)}_{\text{PM}}+1\right)^2\ r'\left((\breve{\theta}^{(j-1)}_{\text{PM}}+1)T\right) - \left(\breve{\theta}^{(j)}_{\text{PM}}\right)^2\ r'\left(\breve{\theta}^{(j)}_{\text{PM}}T\right)\Bigg\}\\ &+\left(\breve{\theta}^{(n)}_{\text{PM}}-n\right)^2\ r'\left(\breve{\theta}^{(n)}_{\text{PM}}T+L-nT\right) \geq 0.\end{aligned} \tag{4.14}$$

Since $\left(\breve{\theta}^{(n)}_{\text{PM}}-n\right) \leq 0$ (see equation (4.5)), we have $\left(\breve{\theta}^{(n)}_{\text{PM}}-n\right)^2 \geq 0$. Furthermore, $r(\cdot)$ is increasing and continuously differentiable, thus $r'(\cdot) \geq 0$. Therefore, the last term in (4.14) satisfies,

$$\left(\breve{\theta}^{(n)}_{\text{PM}}-n\right)^2 r'\left(\breve{\theta}^{(n)}_{\text{PM}}T+L-nT\right) \geq 0.$$

Hence, to show (4.14) it suffices to have

$$\sum_{j=1}^{n}\left\{\left(\breve{\theta}_{\text{PM}}^{(j-1)}+1\right)^{2}\ r'\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-\left(\breve{\theta}_{\text{PM}}^{(j)}\right)^{2}\ r'\left(\breve{\theta}_{\text{PM}}^{(j)}T\right)\right\}\geq 0. \tag{4.15}$$

Consider the j^{th} term in inequality (4.15) and note that it suffices to show

$$\left(\breve{\theta}_{\text{PM}}^{(j-1)}+1\right)^{2}\ r'\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-\left(\breve{\theta}_{\text{PM}}^{(j)}\right)^{2}\ r'\left(\breve{\theta}_{\text{PM}}^{(j)}T\right)\geq 0,$$

or equivalently

$$r'\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)\geq\left(\frac{\breve{\theta}_{\text{PM}}^{(j)}}{\breve{\theta}_{\text{PM}}^{(j-1)}+1}\right)^{2}\ r'\left(\breve{\theta}_{\text{PM}}^{(j)}T\right),$$

which is true since $\left(\frac{\breve{\theta}_{\text{PM}}^{(j)}}{\breve{\theta}_{\text{PM}}^{(j-1)}+1}\right)=\theta_{\text{PM}}^{j}$ (see equation (4.6)), hence $\Lambda\left(L;n,T\right)$ is convex in T. □

In practice, estimates of θ_{PM}^{j} are obtained from the equipment maintenance history, and we do not expect the sufficient condition (4.13) from Proposition 4.1.2.1 to hold for general values of θ_{PM}^{j}. Therefore, we seek conditions on $r(\cdot)$ under which condition (4.13) holds independent of the specific values of θ_{PM}^{j}. The following corollary lists two such conditions.

Corollary 4.1.2.1 *If $r(\cdot)$ is increasing and continuously differentiable then for a fixed $n\in\mathbb{N}$, $\Lambda(L;n,T)$ is convex in T for $T\in\left[0,\frac{L}{n}\right]$ provided, either*

(i) $r(\cdot)$ is convex, or

(ii) $r(\cdot)$ follows a Weibull failure rate.

Proof: In turn, we show that condition (4.13) holds under (i) and under (ii).
(i) If $r(\cdot)$ is convex and continuously differentiable then

$$r'(s)\geq r'(t)\quad\forall\ s\geq t. \tag{4.16}$$

Since $\left(\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1\right) \geq \breve{\theta}_{\mathrm{PM}}^{(j)}$ for all $j=1,2,\ldots,n$ (see equation (4.7)) and $T \geq 0$, we have

$$
\begin{aligned}
& \left(\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1\right) T \geq \breve{\theta}_{\mathrm{PM}}^{(j)} T \\
\Rightarrow \quad & r'\left(\left(\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1\right) T\right) \geq r'\left(\breve{\theta}_{\mathrm{PM}}^{(j)} T\right) \qquad \text{from (4.16)} \\
\Rightarrow \quad & r'\left(\left(\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1\right) T\right) \geq \left(\theta_{\mathrm{PM}}^{j}\right)^{2} \; r'\left(\breve{\theta}_{\mathrm{PM}}^{(j)} T\right) \quad \text{since } \theta_{\mathrm{PM}}^{j} \in [0,1].
\end{aligned}
$$

(ii) If $r(\cdot)$ follows an increasing Weibull failure rate with the *shape* and *scale* parameters $\theta_1 > 1$ and $\theta_2 > 0$, respectively, then

$$
r'(t) = \frac{\theta_1(\theta_1 - 1)}{(\theta_2)^{\theta_1}} t^{\theta_1 - 2}.
$$

Therefore, condition (4.13) holds, if for all $j = 1, 2, \ldots, n$,

$$
\frac{\theta_1(\theta_1 - 1)}{(\theta_2)^{\theta_1}} \left(\left(\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1\right) T\right)^{\theta_1 - 2} \geq \; \left(\theta_{\mathrm{PM}}^{j}\right)^{2} \frac{\theta_1(\theta_1 - 1)}{(\theta_2)^{\theta_1}} \left(\breve{\theta}_{\mathrm{PM}}^{(j)} T\right)^{\theta_1 - 2}.
$$

Because $\theta_1 > 1$, it suffices to show

$$
\left(\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1\right)^{\theta_1 - 2} \geq \left(\theta_{\mathrm{PM}}^{j}\right)^{2} \left(\breve{\theta}_{\mathrm{PM}}^{(j)}\right)^{\theta_1 - 2},
$$

or equivalently

$$
\left(\theta_{\mathrm{PM}}^{j}\right)^{2} \left(\frac{\breve{\theta}_{\mathrm{PM}}^{(j)}}{\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1}\right)^{\theta_1 - 2} \leq 1.
$$

From equation (4.6), we have $\left(\dfrac{\breve{\theta}_{\mathrm{PM}}^{(j)}}{\breve{\theta}_{\mathrm{PM}}^{(j-1)}+1}\right) = \theta_{\mathrm{PM}}^{j}$ and hence it suffices to have

$$
\left(\theta_{\mathrm{PM}}^{j}\right)^{2} \left(\theta_{\mathrm{PM}}^{j}\right)^{\theta_1 - 2} = \left(\theta_{\mathrm{PM}}^{j}\right)^{\theta_1} \leq 1,
$$

which holds since $\theta_{\mathrm{PM}}^{j} \in [0,1], \quad \forall j = 1, 2, \ldots, n.$ $\square$

Corollary 4.1.2.1 brings forward a key set of results. Recall that an increasing failure simply indicates that the condition of a repairable unit of equipment deteriorates with its increasing *virtual* age. In practice, an increasing Weibull failure rate is by far the most commonly-used assumption in the reliability literature. In addition, an increasing and convex failure rate implies that the *rate* at which condition of the equipment deteriorates, also increases with its *virtual* age, which is true for most equipment. Therefore, an increasing and convex failure rate is often a reasonable assumption in practice. Under both these assumptions, the specific numerical values of the PM age reduction factors have no bearing on the convexity of $\Lambda(L; n, T)$, in T, for a fixed value of n. In the following proposition, we extend the convexity results to $\Lambda_2(L; n, T), \bar{\Lambda}_\alpha(L; n, T)$ and $\bar{C}_\alpha(L; n, T)$.

Proposition 4.1.2.2 *If CMs are minimal repairs, and $\Lambda(L; n, T)$ is convex and continuously differentiable in T, then for a fixed $n \in \mathbb{N}$ and L,*

(*i*) *$\Lambda_2(L; n, T), \bar{\Lambda}_\alpha(L; n, T)$, and $\bar{C}_\alpha(L; n, T)$ are also convex in T, for $T \in [0, \frac{L}{n}]$; and,*

(*ii*) *the solution to the following equation simultaneously minimizes $\Lambda(L; n, T)$, $\Lambda_2(L; n, T)$, $\bar{\Lambda}_\alpha(L; n, T)$ and $\bar{C}_\alpha(L; n, T)$:*

$$\sum_{j=1}^{n} \left\{ \left(\breve{\theta}_{PM}^{(j-1)} + 1 \right) \ r\left((\breve{\theta}_{PM}^{(j-1)} + 1)T \right) - \breve{\theta}_{PM}^{(j)} \ r\left(\breve{\theta}_{PM}^{(j)} T \right) \right\}$$
$$= \left(n - \breve{\theta}_{PM}^{(n)} \right) \ r\left(\breve{\theta}_{PM}^{(n)} T + L - nT \right). \tag{4.17}$$

Proof: (*i*) For a fixed value of $n \in \mathbb{N}$, $\Lambda(L; n, T)$, $\Lambda_2(L; n, T), \bar{\Lambda}_\alpha(L; n, T)$, and $\bar{C}_\alpha(L; n, T)$ are defined for $T \in \left[0, \frac{L}{n}\right]$. Furthermore, if CMs are modeled as minimal repairs and $\Lambda(L; n, T)$ is convex in T, then from equation (4.9), $\Lambda_2(L; n, T)$ is the sum of two convex functions, therefore it is convex in T. Similarly, $\bar{\Lambda}_\alpha(L;n,T)$ and $\bar{C}_\alpha(L;n,T)$ in equations (4.10) and (4.11), respectively, are also the sum of two convex functions, hence both $\bar{\Lambda}_\alpha(L;n,T)$ and $\bar{C}_\alpha(L;n,T)$ are convex in T, for a fixed value of n.

(ii) Suppose T minimizes $\bar{C}_\alpha(L;n,T)$. Then, the convexity result for $\bar{C}_\alpha(L;n,T)$ from part *(i)* implies that

$$\frac{\partial \bar{C}_\alpha\left(L;n,T\right)}{\partial T}=0, \tag{4.18}$$

is the necessary and sufficient condition for T to be an optimal solution to the second-stage optimization problem. From equations (4.10), (4.11) and (4.18), we have

$$\frac{\partial}{\partial T}\left(nC_{\text{PM}}+\bar{\Lambda}_\alpha(L;n,T)C_{\text{CM}}\right)=0 \quad\Leftrightarrow\quad \frac{\partial\bar{\Lambda}_\alpha(L;n,T)}{\partial T}=0 \tag{4.19}$$

$$\Leftrightarrow\quad \left(1+\frac{\alpha}{2}\right)\frac{\partial\Lambda(L;n,T)}{\partial T}+\frac{\alpha}{2}\frac{\partial\Lambda_2(L;n,T)}{\partial T}=0. \tag{4.20}$$

From equation (4.9) this is equivalent to:

$$\left(1+\alpha+\alpha\Lambda(L;n,T)\right)\frac{\partial\Lambda(L;n,T)}{\partial T}=0 \quad\Leftrightarrow\quad \frac{\partial\Lambda(L;n,T)}{\partial T}=0. \tag{4.21}$$

Using (4.8) we have that (4.21) is equivalent to:

$$\begin{aligned}
&\sum_{j=1}^{n}\left\{\left(\breve{\theta}_{\text{PM}}^{(j-1)}+1\right)\ r\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-\breve{\theta}_{\text{PM}}^{(j-1)}\ r\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\}\\
&\qquad+\left(\breve{\theta}_{\text{PM}}^{(n)}-n\right)\ r\left(\breve{\theta}_{\text{PM}}^{(n)}T+L-nT\right)-\breve{\theta}_{\text{PM}}^{(n)}\ r\left(\breve{\theta}_{\text{PM}}^{(n)}T\right)=0\\
\Leftrightarrow\quad&\sum_{j=1}^{n}\left\{\left(\breve{\theta}_{\text{PM}}^{(j-1)}+1\right)\ r\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-\breve{\theta}_{\text{PM}}^{(j)}\ r\left(\breve{\theta}_{\text{PM}}^{(j)}T\right)\right\}\\
&\qquad=\left(n-\breve{\theta}_{\text{PM}}^{(n)}\right)\ r\left(\breve{\theta}_{\text{PM}}^{(n)}T+L-nT\right),
\end{aligned}$$

where the final equivalence holds because $\breve{\theta}^{(0)}_{\text{PM}} = 0$. To complete the proof, we note that the necessary and sufficient conditions that T minimizes $\bar{\Lambda}_\alpha(L; n, T), \Lambda_2(L; n, T)$ and $\Lambda(L; n, T)$, appear in equations (4.19), (4.20) and (4.21), respectively, all of which lead to equation (4.17). □

Given that $\Lambda(L; n, T)$ is convex T, the second-stage optimization problem from (4.12), can be efficiently solved using well-known line search methods, such as, a golden section search or Fibonacci search (see, e.g., Luenberger 1984). Furthermore, the two-stage optimization problem can be solved by comparing the optimal solutions to the second-stage problems, for $n = 0, 1, \ldots, n_{\max}$, where, $n_{\max}$ represents the maximum number of PMs that can possibly be scheduled over the finite planning horizon, $[0, L]$. Alternatively an $n_{\max}$ value can be derived computationally. Let $\bar{C}_\alpha(L; \hat{n}, T^*_{\hat{n}})$ be the expected total cost for some $\hat{n}$ where $T^*_{\hat{n}} \in \arg\min_{T \in [0, \frac{L}{\hat{n}}]} \bar{C}_\alpha(L; \hat{n}, T)$. Then, we can take $n_{\max} = \left\lfloor \dfrac{\bar{C}_\alpha(L; \hat{n}, T^*_{\hat{n}})}{C_{\text{PM}}} \right\rfloor$.

4.2 Periodic PM Optimization Models under $\mathcal{M}(1)$

As discussed in Section 3.1, most of the reliability literature for optimizing PM over a finite planning horizon restricts the class of PM policies to have equal-length PM intervals, i.e., $T = \frac{L}{n}$, $n \in \mathbb{N}$ (see Nakagawa & Mizutani 2009, for more details). In the previous section, we formulate a two-stage model in which T and n are selected separately with $T \in \left[0, \frac{L}{n}\right]$. This allows the last PM interval to have a different length of $L - nT$. Boland & Proschan (1982) and then Galenko et al. (2005) also allow the final PM interval to have a different length but their formulations implicitly assume that the last PM interval is shorter than the others. Here, we recall their class of PM optimization models, which we call $\mathcal{M}(1)$, from Section 3.1, except that our models also allow for age reduction factors. In this section, we characterize solutions to $\mathcal{M}(1)$ in an analogous fashion to what we did for $\mathcal{M}(2)$ in Section 4.1, including convexity results that carry forward. Finally, we establish that $\mathcal{M}(1)$ is indeed a restriction of $\mathcal{M}(2)$ in general, but we provide conditions under which these two classes of models are equivalent.

We recall the generalized formulation under our first class of periodic PM optimization models, $\mathcal{M}(1)$, from Section 3.1, and write

$$\begin{aligned} z^*(L) &= \min_{T\in[0,L]} \mathbb{E}\left[\mathcal{C}\left(L;T\right)\right] \\ &= \min_{T\in[0,L]} \left\{\mathbb{E}\left[\mathcal{C}_p(L;T)\right] + \mathbb{E}\left[\mathcal{C}_c(L;T)\right]\right\}. \end{aligned} \tag{4.22}$$

As in Section 3.1, the optimal periodic maintenance policy under $\mathcal{M}(1)$, aims at minimizing the expected value of a random total cost function, $\mathcal{C}\left(L;T\right)$, which is the sum of two random cost functions, $\mathcal{C}_p(L;T)$ and $\mathcal{C}_c(L;T)$. The former denotes the random cost of all the PMs and the latter defines the random CM cost of $\mathcal{N}(L;T)$ failures over $[0,L]$.

4.2.1 Risk-Averse Formulation

For continuity, we re-write the risk-averse formulation in equation (4.22), under the previously used notion of *global* risk. We use the definition of the CM cost function, $\mathcal{C}_{\text{CM}}(\cdot)$, from equation (3.8). Therefore, we have

$$\begin{aligned} \mathbb{E}\left[\mathcal{C}_c(L;T)\right] &= \mathbb{E}\left[\mathcal{C}_{\text{CM}}\left(\mathcal{N}\left(L;T\right)\right)\right] \\ &= \mathbb{E}\left[\left(\left(1+\frac{\alpha}{2}\right)\mathcal{N}(L;T) + \frac{\alpha}{2}\mathcal{N}(L;T)^2\right)\mathcal{C}_{\text{CM}}\right] \\ &= \left(\left(1+\frac{\alpha}{2}\right)\Lambda(L;T) + \frac{\alpha}{2}\Lambda_2(L;T)\right)C_{\text{CM}}. \end{aligned} \tag{4.23}$$

Furthermore, we assume that a PM is not required at time L, and use equations (4.22) and (4.23) to write the risk-averse formulation in equation (4.22) as,

$$z^*(L) = \min_{T\in[0,L]} \left\{ \left(\left\lceil \frac{L}{T} \right\rceil - 1 \right) C_{\mathrm{PM}} + \left(\left(1 + \frac{\alpha}{2}\right) \Lambda(L;T) + \frac{\alpha}{2}\Lambda_2(L;T) \right) C_{\mathrm{CM}} \right\}. \tag{4.24}$$

4.2.2 Imperfect PMs and Minimal CMs

Our minimal CM assumption leads to a closed-form expression for the expected number of equipment failures, $\Lambda(L;T)$, similar to the expression for $\Lambda(L;n,T)$, provided in equation (4.8). The only difference here is that for a given value of $T \in [0, L]$, there are $\lfloor \frac{L}{T} \rfloor$ PM intervals of length T, followed by a possible interval of length $L - \lfloor \frac{L}{T} \rfloor T$. Therefore, for general values of PM age reduction factors, $\theta^j_{PM}, j \in \mathbb{N}$, we have

$$\begin{aligned}
\Lambda(L;T) &= \sum_{j=1}^{\lfloor \frac{L}{T} \rfloor} \Big\{ R\Big(v\left((j-1)T\right) + T\Big) - R\Big(v\left((j-1)T\right)\Big) \Big\} \\
&\quad + R\left(v\left(\left\lfloor \frac{L}{T} \right\rfloor T\right) + L - \left\lfloor \frac{L}{T} \right\rfloor T\right) - R\left(v\left(\left\lfloor \frac{L}{T} \right\rfloor T\right)\right) \\
&= \sum_{j=1}^{\lfloor \frac{L}{T} \rfloor} \left\{ R\left((\breve{\theta}^{(j-1)}_{\mathrm{PM}} + 1)T\right) - R\left(\breve{\theta}^{(j-1)}_{\mathrm{PM}} T\right) \right\} \\
&\quad + R\left(\breve{\theta}^{(\lfloor \frac{L}{T} \rfloor)}_{\mathrm{PM}} T + L - \left\lfloor \frac{L}{T} \right\rfloor T\right) - R\left(\breve{\theta}^{(\lfloor \frac{L}{T} \rfloor)}_{\mathrm{PM}} T\right),
\end{aligned} \tag{4.25}$$

where, $R(t) = \displaystyle\int_0^t r(u)\mathrm{d}u$, and $\breve{\theta}^{(\cdot)}_{\mathrm{PM}}$ is defined in equation (4.5). Furthermore, under the minimal CM assumption,

$$\Lambda_2(L;T) = \Lambda(L;T)\left(\Lambda(L;T) + 1\right). \tag{4.26}$$

For notational convenience, we write

$$\bar{\Lambda}_\alpha(L;T) = \left(1+\frac{\alpha}{2}\right)\Lambda(L;T) + \frac{\alpha}{2}\Lambda_2(L;T), \tag{4.27}$$

$$\begin{aligned}\bar{C}_\alpha(L;T) &= \left(\left\lceil \frac{L}{T}\right\rceil - 1\right)C_{\text{PM}} + \left(\left(1+\frac{\alpha}{2}\right)\Lambda(L;T) + \frac{\alpha}{2}\Lambda_2(L;T)\right)C_{\text{CM}} \\ &= \left(\left\lceil \frac{L}{T}\right\rceil - 1\right)C_{\text{PM}} + \bar{\Lambda}_\alpha(L;T)C_{\text{CM}}, \end{aligned} \tag{4.28}$$

and represent the risk-averse formulation of the periodic PM optimization problem from equation (4.24) as,

$$z^*(L) = \min_{T\in[0,L]} \bar{C}_\alpha(L;T). \tag{4.29}$$

We note that the formulation in equation (4.29) differs from the two-stage periodic PM model presented in equation (4.12). For convenience, we refer to the optimization problem in equation (4.29), as a single-stage periodic PM model. Recall that the two-stage model has two decision variables, n and T, while T is the only decision variable in the single-stage model. In the two-stage model, we first fix the number of PMs n and then select $T \in [0, \frac{L}{n}]$, whereas, in the single-stage model, our selection of $T \in [0, L]$ automatically fixes the number of PMs to $\left\lceil \frac{L}{T} \right\rceil - 1$. Furthermore, in the two-stage model, for a fixed $n \in \mathbb{N}$, there are n PM intervals of length T, followed by a possible interval of length $L - nT \geq 0$. However, in the single-stage model, for a given value of $T \in [0, L]$, there are $\left\lfloor \frac{L}{T} \right\rfloor$ PM intervals of length T, followed by a possible interval of length $0 \leq L - \left\lfloor \frac{L}{T} \right\rfloor T \leq T$. Note that this last interval in the single-stage model, if $\frac{L}{T}$ is not integer, has length at most T, whereas, this length can be greater than T in the more general two-stage model.

For a fixed value of n, $\Lambda(L;n,T)$ defined in equation (4.8) is continuous in T. However, the presence of the floor operator in the definition of $\Lambda(L;T)$ (see equation (4.25)), at first, raises a concern on the continuity of $\Lambda(L;T)$. The following proposition addresses this concern by showing that $\Lambda(L;T)$ is continuous in T, and also presents continuity results for $\Lambda_2(L;T)$, $\bar{\Lambda}_\alpha(L;T)$ and $\bar{C}_\alpha(L;T)$.

Proposition 4.2.2.1 *If CMs are minimal repairs then for general values of the PM age reduction factors, $\theta^j_{PM}, j \in \mathbb{N}$,*

(i) $\Lambda(L;T)$ *is continuous in* T;

(ii) $\Lambda_2(L;T)$ *and* $\bar{\Lambda}_\alpha(L;T)$ *are continuous in* T; *and,*

(iii) $\bar{C}_\alpha(L;T)$ *is lower semicontinuous in* T *with the set of discontinuities,* $\mathbb{D} = \left\{d : d = \frac{L}{n}, n \in \mathbb{N}\right\}$.

Proof: *(i)* We first note that $\left\lfloor \frac{L}{T} \right\rfloor$ is left-continuous in T, with points of discontinuity given by the set, $\mathbb{D} = \left\{d : d = \frac{L}{n}, n \in \mathbb{N}\right\}$. Then,

$$\lim_{T \to \frac{L}{n+1}^-} \left\lfloor \frac{L}{T} \right\rfloor = n+1, \text{ and } \lim_{T \to \frac{L}{n+1}^+} \left\lfloor \frac{L}{T} \right\rfloor = n. \tag{4.30}$$

Now $\Lambda(L;\ \cdot)$ is continuous at T if

$$\lim_{T \to \frac{L}{n+1}^-} \Lambda(L;T) = \lim_{T \to \frac{L}{n+1}^+} \Lambda(L;T).$$

Substituting $\Lambda(L;T)$ from equation (4.25), it suffices to show

$$\begin{aligned}
&\lim_{T \to \frac{L}{n+1}^-} \sum_{j=1}^{\left\lfloor \frac{L}{T} \right\rfloor} \left\{ R\left((\breve{\theta}^{(j-1)}_{\text{PM}} + 1)T \right) - R\left(\breve{\theta}^{(j-1)}_{\text{PM}} T \right) \right\} \\
&\quad + R\left(\breve{\theta}^{\left(\left\lfloor \frac{L}{T} \right\rfloor\right)}_{\text{PM}} T + L - \left\lfloor \frac{L}{T} \right\rfloor T \right) - R\left(\breve{\theta}^{\left(\left\lfloor \frac{L}{T} \right\rfloor\right)}_{\text{PM}} T \right). \\
&= \lim_{T \to \frac{L}{n+1}^+} \sum_{j=1}^{\left\lfloor \frac{L}{T} \right\rfloor} \left\{ R\left((\breve{\theta}^{(j-1)}_{\text{PM}} + 1)T \right) - R\left(\breve{\theta}^{(j-1)}_{\text{PM}} T \right) \right\} \\
&\quad + R\left(\breve{\theta}^{\left(\left\lfloor \frac{L}{T} \right\rfloor\right)}_{\text{PM}} T + L - \left\lfloor \frac{L}{T} \right\rfloor T \right) - R\left(\breve{\theta}^{\left(\left\lfloor \frac{L}{T} \right\rfloor\right)}_{\text{PM}} T \right)
\end{aligned}$$

Using (4.30), it suffices to have

$$\begin{aligned}
&\sum_{j=1}^{n+1}\left\{R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\}\\
&\qquad+R\left(\breve{\theta}_{\text{PM}}^{(n+1)}T+L-(n+1)T\right)-R\left(\breve{\theta}_{\text{PM}}^{(n+1)}T\right)\\
&\quad=\sum_{j=1}^{n}\left\{R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\}\\
&\qquad+R\left(\breve{\theta}_{\text{PM}}^{(n)}T+L-nT\right)-R\left(\breve{\theta}_{\text{PM}}^{(n)}T\right).
\end{aligned}$$

Since $L=(n+1)T$, it suffices to show

$$\begin{aligned}
&\sum_{j=1}^{n+1}\left\{R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\}\\
&\quad=\sum_{j=1}^{n}\left\{R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\}\\
&\qquad+R\left(\breve{\theta}_{\text{PM}}^{(n)}T+T\right)-R\left(\breve{\theta}_{\text{PM}}^{(n)}T\right),
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\sum_{j=1}^{n+1}\left\{R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\}\\
&\quad=\sum_{j=1}^{n+1}\left\{R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right)-R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\}.
\end{aligned}$$

Which is true, hence $\Lambda(L;T)$ is continuous in T.

(ii) Part *(i)* of this proposition establishes that $\Lambda(L;T)$ is continuous in T. From equation (4.26), $\Lambda_2(L;T)=\Lambda(L;T)^2+\Lambda(L;T)$, which is the sum of two continuous functions, and therefore, $\Lambda_2(L;T)$ is continuous in T. Similarly, $\bar{\Lambda}_\alpha(L;T)$ in (4.27) is also the sum of two continuous functions, and hence $\bar{\Lambda}_\alpha(L;T)$ is continuous in T.

(iii) We note that because of the ceiling operator, the first term in $\bar{C}_\alpha(L;T)$, $\left(\lceil \frac{L}{T} \rceil - 1\right) C_{\text{PM}}$ is lower semicontinuous in T, with points of discontinuity given by the set, $\mathbb{D}$. Furthermore, part (ii) of this proposition suffices to show the continuity of the second term, $\bar{\Lambda}_\alpha(L;T)C_{\text{CM}}$. Therefore, $\bar{C}_\alpha(L;T)$ is lower semicontinuous in T. $\square$

We recall that Corollary 4.1.2.1 establishes convexity of $\Lambda(L;n,T)$, which leads to the convexity of $\Lambda_2(L;n,T)$, $\Lambda_\alpha(L;n,T)$ and $\bar{C}_\alpha(L;n,T)$, for a wide class of failure rate functions. These convexity results are shown for a fixed value of n, and we note that the notion of a fixed number of PMs, n, in the two-stage model is equivalent to $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$ in the single-stage model. Now that we have obtained continuity results for $\Lambda(L;T)$, $\Lambda_2(L;T)$, $\Lambda_\alpha(L;T)$ and $\bar{C}_\alpha(L;T)$, we seek connections between the two models. The following proposition establishes the relationship between the single-stage and the two-stage periodic PM models.

Proposition 4.2.2.2 *If CMs are minimal repairs then for general values of PM age reduction factors,* $\theta^j_{PM}, j \in \mathbb{N}$, *and a fixed value of* $n \in \mathbb{N}$,

(i) $\Lambda(L;n,T) = \Lambda(L;T), \qquad T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$;

(ii) $\Lambda_2(L;n,T) = \Lambda_2(L;T)$ *and* $\bar{\Lambda}_\alpha(L;n,T) = \bar{\Lambda}_\alpha(L;T), \qquad T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$; *and,*

(iii) $\bar{C}_\alpha(L;n,T) = \bar{C}_\alpha(L;T), \qquad T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$.

In addition, if $\Lambda(L;n,T)$ *is convex in* T *for a fixed* n *and* $T \in \left[0, \frac{L}{n}\right]$ *then,*

(iv) $\Lambda(L;T)$, $\Lambda_2(L;T)$, *and* $\bar{\Lambda}_\alpha(L;T)$ *are convex in* T *for* $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$; *and,*

(v) $\bar{C}_\alpha(L;T)$ *is convex in* T *for* $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$.

Proof: (i) We recall that in the two-stage periodic PM optimization model with imperfect PMs and minimal CMs, is given by

$$\Lambda(L;n,T) = \sum_{j=1}^{n} \left\{ R\left(\left(\breve{\theta}_{\text{PM}}^{(j-1)}+1\right)T\right) - R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\} + R\left(\breve{\theta}_{\text{PM}}^{(n)}T + L - nT\right) - R\left(\breve{\theta}_{\text{PM}}^{(n)}T\right).$$

Similarly, the expected number of equipment failures, $\Lambda(L;T)$, in the single-stage model, is given by

$$\Lambda(L;T) = \sum_{j=1}^{\lfloor \frac{L}{T} \rfloor} \left\{ R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right) - R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\} + R\left(\breve{\theta}_{\text{PM}}^{(\lfloor \frac{L}{T} \rfloor)}T + L - \left\lfloor \frac{L}{T} \right\rfloor T\right) - R\left(\breve{\theta}_{\text{PM}}^{(\lfloor \frac{L}{T} \rfloor)}T\right).$$

For $T \in \left(\frac{L}{n+1}, \frac{L}{n}\right]$, $\left\lfloor \frac{L}{T} \right\rfloor = n$, therefore $\Lambda(L;n,T) = \Lambda(L;T)$. Furthermore, at $T = \frac{L}{n+1}$, $\left\lfloor \frac{L}{T} \right\rfloor = n+1$ and we have,

$$\begin{aligned}
\Lambda(L;n,T)|_{L=(n+1)T} &= \sum_{j=1}^{n} \left\{ R\left(\left(\breve{\theta}_{\text{PM}}^{(j-1)}+1\right)T\right) - R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\} \\
&\quad + R\left(\breve{\theta}_{\text{PM}}^{(n)}T + (n+1)T - nT\right) - R\left(\breve{\theta}_{\text{PM}}^{(n)}T\right) \\
&= \sum_{j=1}^{n+1} \left\{ R\left((\breve{\theta}_{\text{PM}}^{(j-1)}+1)T\right) - R\left(\breve{\theta}_{\text{PM}}^{(j-1)}T\right)\right\} \\
&= \Lambda(L;T)|_{L=(n+1)T}\,.
\end{aligned}$$

This completes the proof of part (i).

(ii) If CMs are minimal then for a fixed $n \in \mathbb{N}$,

$$\begin{aligned}
\Lambda_2(L;n,T) &= \Lambda(L;n,T)\left(\Lambda(L;n,T)+1\right) \\
&= \Lambda(L;T)\left(\Lambda(L;T)+1\right) \qquad \text{for } T \in \left[\frac{L}{n+1}, \frac{L}{n}\right] \\
&= \Lambda_2(L;T),
\end{aligned}$$

where the second equality holds from part (i).

Similarly,

$$\begin{aligned}\bar{\Lambda}_\alpha(L;n,T) &= \left(1+\frac{\alpha}{2}\right)\Lambda(L;n,T) + \frac{\alpha}{2}\Lambda_2(L;n,T)\\ &= \left(1+\frac{\alpha}{2}\right)\Lambda(L;T) + \frac{\alpha}{2}\Lambda_2(L;T) \qquad \text{for } T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]\\ &= \bar{\Lambda}_\alpha(L;T).\end{aligned}$$

(iii) We note that $\lceil \frac{L}{T} \rceil = n+1$ for $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$, and therefore

$$\begin{aligned}\bar{C}_\alpha(L;T) &= \left(\left\lceil \frac{L}{T} \right\rceil - 1\right) C_{\text{PM}} + \bar{\Lambda}_\alpha(L;T) C_{\text{CM}}\\ &= nC_{\text{PM}} + \bar{\Lambda}_\alpha(L;T) C_{\text{CM}}\\ &= nC_{\text{PM}} + \bar{\Lambda}_\alpha(L;n,T) C_{\text{CM}}\\ &= \bar{C}_\alpha(L;n,T) \qquad \text{for } T \in \left[\frac{L}{n+1}, \frac{L}{n}\right),\end{aligned}$$

where the third equality holds from part *(ii)*.

(iv) Given that $\Lambda(L;n,T)$ is convex in T for a fixed n, and $T \in \left[0, \frac{L}{n}\right]$, part *(i)* of Proposition 4.1.2.2 shows that $\Lambda_2(L;n,T)$ and $\bar{\Lambda}_\alpha(L;n,T)$ are also convex in T. Therefore, convexity of $\Lambda(L;T)$, $\Lambda_2(L;T)$ and $\bar{\Lambda}_\alpha(L;T)$ in T for $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$ follow from the equality results established in parts *(i)* and *(ii)* of this proposition.

(v) Convexity of $\Lambda(L;n,T)$ in T for a fixed n, leads to convexity of $\bar{C}_\alpha(L;n,T)$ in T for $T \in \left[0, \frac{L}{n}\right]$; see part *(i)* of Proposition 4.1.2.2. Furthermore, part *(iii)* of this proposition establishes that $\bar{C}_\alpha(L;n,T) = \bar{C}_\alpha(L;T)$ for $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$, which suffices to show the convexity of $\bar{C}_\alpha(L;T)$. □

Figure 4.1 summarizes the key theoretical results from Corollary 4.1.2.1, and Propositions 4.1.2.2, 4.2.2.1 and 4.2.2.2. We fix $L = 100$, $\theta^j_{\text{PM}} = 0.25$ for all $j \in \mathbb{N}$, and consider a Weibull failure rate with *shape* parameter, $\theta_1 = 1.25$, and *scale*

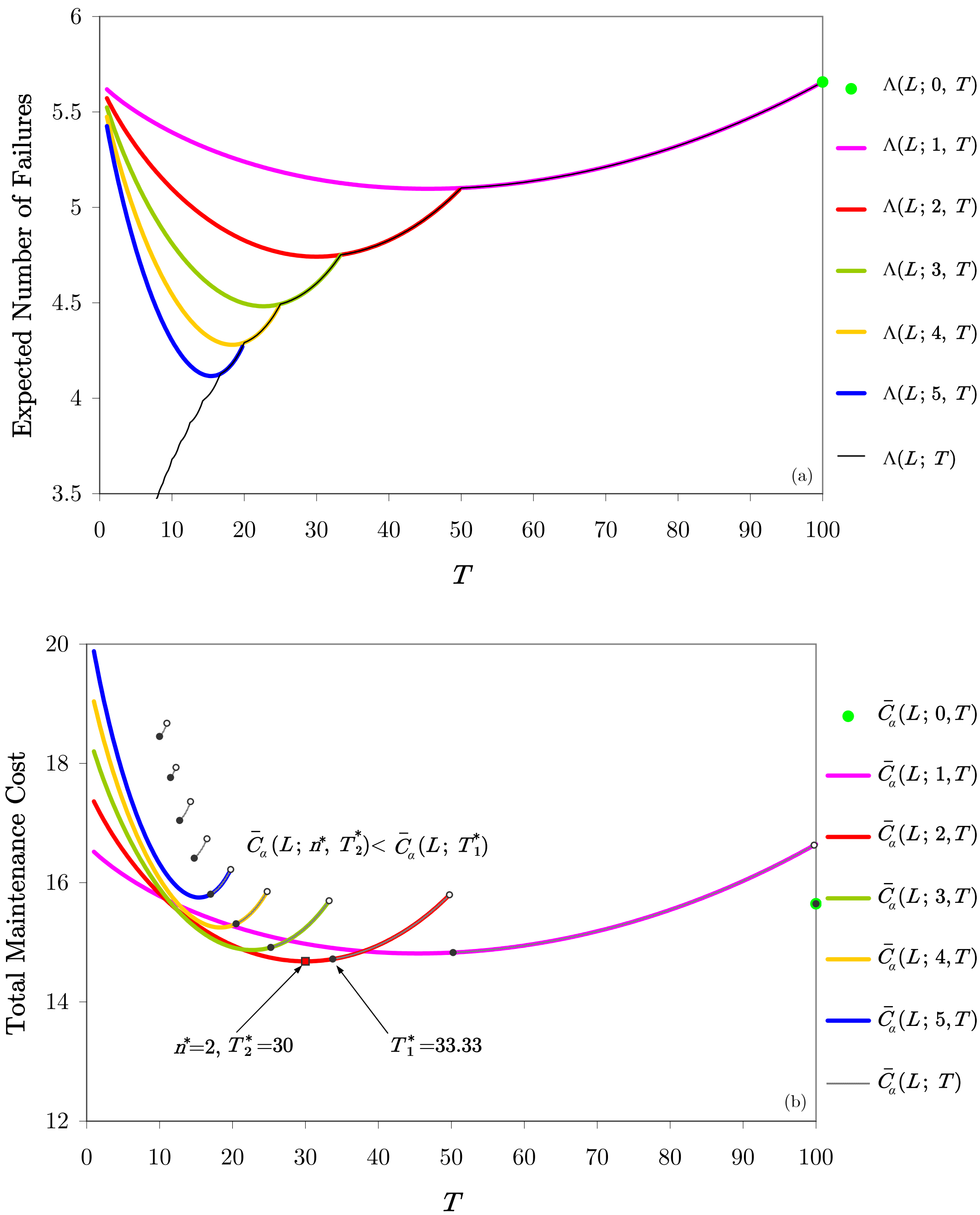

Figure 4.1: Comparison between (a) $\Lambda(L;T)$ and $\Lambda(L;n,T)$, and (b) $\bar{C}_\alpha(L;T)$ and $\bar{C}_\alpha(L;n,T)$, from single-stage and two-stage periodic PM optimization models, under imperfect PMs and minimal CMs.

parameter, $\theta_2 = 25$. Since $1 < \theta_1 < 2$, the failure rate is increasing but it is not convex. Figure 4.1(a) plots $\Lambda(L; n, T)$ as a function of T, for $n = 0, 1, \ldots, 5$. We recall that, in the two-stage model, $T = L$ is the only feasible solution for $n = 0$, therefore, it is represented as a point. For a fixed value of n, Figure 4.1(a) shows that $\Lambda(L; n, T)$ is convex in T, which is consistent with part (ii) of Corollary 4.1.2.1. Figure 4.1(a) also shows that $\Lambda(L; T)$ from the single-stage periodic PM model is continuous in T (see part (i) of Proposition 4.2.2.1) and for a given value of n, it equals $\Lambda(L; n, T)$, $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$ (refer to part (i) of Proposition 4.2.2.2).

We fix $C_{\text{PM}} = 1$, $C_{\text{CM}} = 2$, and $\alpha = 10\%$, and compute the total expected maintenance cost. For a fixed value of n, Figure 4.1(b) shows that $\bar{C}_\alpha(L; n, T)$ is convex in T, which is consistent with part (i) of Proposition 4.1.2.2. Furthermore, Figure 4.1(b) shows that $\bar{C}_\alpha(L; T)$ is lower-semicontinuous (refer to part (iii) of Proposition 4.2.2.1), and for a given value of n, it agrees with $\bar{C}_\alpha(L; n, T)$, $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$ (see part (iii) of Proposition 4.2.2.2). Finally, we note that the two periodic PM models, provide different optimal solutions. The optimal solution to the single-stage model, $T_1^* = \frac{100}{3}$, suggests $\left\lceil \frac{L}{T_1^*} \right\rceil - 1 = 2$ PMs over $[0, 100]$, which should be performed at $T_1^* = 33.33$, and $2T_2^* = 66.67$. The minimal cost under the single-stage model, $\bar{C}_\alpha(L; T_1^*) = 14.71$. The optimal policy under the two-stage model also advocates $n^* = 2$ PMs, but recommends that the PMs should be performed at $T_2^* = 30$, and $2T_2^* = 60$, over $[0, 100]$. The minimal cost under the two-stage model, $\bar{C}_\alpha(L; n^*, T_2^*) = 14.68$, and therefore, $\bar{C}_\alpha(L; n^*, T_2^*) < \bar{C}_\alpha(L; T_1^*)$. We note that if $\frac{L}{n^*+1} \leq T_2^* < \frac{L}{n^*}$, for some $n^* \in \mathbb{N}_0$ then, $T_1^* = T_2^*$ and $\bar{C}_\alpha(L; n^*, T_2^*) = \bar{C}_\alpha(L; T_1^*)$. The following proposition establishes the relationship between the optimal maintenance costs from the single-stage and the two-stage models.

Proposition 4.2.2.3 *Let (n^*, T^*) be an optimal solution to the two-stage periodic PM optimization problem with minimal CMs and imperfect PMs. Then, one of the following must hold:*

(i) *If $\frac{L}{n^*+1} \leq T^* \leq \frac{L}{n^*}$ for $n^* \in \mathbb{N}_0$ then*

$$\min_{n \in \mathbb{N}_0} \left\{ \min_{T \in \left[0, \frac{L}{n}\right]} \bar{C}_\alpha(L; n, T) \right\} = \min_{T \in [0,L]} \bar{C}_\alpha(L; T); \tag{4.31}$$

(ii) *otherwise, $0 \leq T^* < \frac{L}{n^*+1}$ for $n^* \in \mathbb{N}_0$ and*

$$\min_{n \in \mathbb{N}_0} \left\{ \min_{T \in \left[0, \frac{L}{n}\right]} \bar{C}_\alpha(L; n, T) \right\} \leq \min_{T \in [0,L]} \bar{C}_\alpha(L; T). \tag{4.32}$$

Proof: Without loss of generality, we can consider $T \in \left[0, \frac{L}{n}\right)$ in the two-stage periodic PM optimization model given in (4.12), because it is straightforward to show that for a given $n^* \in \mathbb{N}$, $T = \frac{L}{n^*}$ cannot be optimal.

(i) We prove that in this case, T^* also optimizes the single-stage periodic PM model, and therefore equation (4.31) holds. We show it by contradiction.
If T^* does not solve the single-stage model then there exists $\hat{T}^* \neq T^*$ satisfying

$$\bar{C}_\alpha(L; T^*) > \bar{C}_\alpha(L; \hat{T}^*). \tag{4.33}$$

Furthermore, there must exist $\hat{n}^*$ such that $\frac{L}{\hat{n}^*+1} \leq \hat{T}^* < \frac{L}{\hat{n}^*}$, for some $\hat{n}^* \in \mathbb{N}_0$. From part (iii) of Proposition 4.2.2.2, we have

$$\bar{C}_\alpha(L; \hat{T}^*) = \bar{C}_\alpha(L; \hat{n}^*, \hat{T}^*). \tag{4.34}$$

Since (n^*, T^*) optimizes the two-stage model and $\frac{L}{n^*+1} \leq T^* < \frac{L}{n^*}$, once again from part (iii) of Proposition 4.2.2.2, we obtain

$$\bar{C}_\alpha(L; n^*, T^*) = \bar{C}_\alpha(L; T^*). \tag{4.35}$$

Equations (4.33), (4.34) and (4.35) lead to,

$$\bar{C}_\alpha(L; n^*, T^*) = \bar{C}_\alpha(L; T^*) > \bar{C}_\alpha(L; \hat{T}^*) = \bar{C}_\alpha(L; \hat{n}^*, \hat{T}^*),$$

which contradicts that (n^*, T^*) is an optimal solution to the two-stage periodic PM optimization problem. Therefore, T^* solves the single-stage periodic PM optimization model and equation (4.31) holds.

(ii) Once again in this case, we show that inequality (4.32) holds by contradiction. Let $\hat{T}^*$ solves the single-stage problem and assume

$$\bar{C}_\alpha(L; n^*, T^*) > \bar{C}_\alpha(L; \hat{T}^*). \tag{4.36}$$

Since $\hat{T}^*$ optimizes the single-stage periodic PM model, there must exist $\hat{n}^*$ such that $\frac{L}{\hat{n}^*+1} \leq \hat{T}^* < \frac{L}{\hat{n}^*}$, for some $\hat{n}^* \in \mathbb{N}_0$. Therefore, from part (iii) of Proposition 4.2.2.2, we have,

$$\bar{C}_\alpha(L; \hat{T}^*) = \bar{C}_\alpha(L; \hat{n}^*, \hat{T}^*). \tag{4.37}$$

From equations (4.36) and (4.37), $\bar{C}_\alpha(L; n^*, T^*) > \bar{C}_\alpha(L; \hat{n}^*, \hat{T}^*)$, which contradicts that (n^*, T^*) minimizes $\bar{C}_\alpha(L; n, T)$, and therefore, equation (4.32) must hold. □

4.2.3 Perfect PMs and Minimal CMs – Idealized View

In Section 4.2.2, we show that the optimal policies from the single-stage and the two-stage periodic PM optimization models can differ. In this section, we establish the equivalence between the two optimization models, under the *idealized view*. We recall that under the *idealized view*, PMs are assumed perfect i.e., $\theta^{j}_{\mathrm{PM}} = 0, \forall j \in \mathbb{N}$, and we have,

$$\breve{\theta}^{(n)}_{\mathrm{PM}} = \sum_{j=1}^{n} \prod_{k=j}^{n} \theta^{k}_{\mathrm{PM}} = 0 \quad \forall n \in \mathbb{N}.$$

Since CMs are modeled as minimal repairs, maintenance models under the *idealized view* are special cases of the imperfect PM models, discussed earlier. Therefore, all of the results presented in Sections 4.1.2 and 4.2.2, apply to two-stage and single-stage periodic PM models, under the *idealized view*, respectively.

In the two-stage model with imperfect PMs and minimal CMs, $\Lambda\left(L; n, T\right)$ is defined in equation (4.8). Substituting $\breve{\theta}^{(j)}_{\mathrm{PM}} = 0$, $j = 1, 2, \ldots, n$, in equation (4.8),

$$\Lambda(L; n, T)_{iv} = nR(T) + R(L - nT), \tag{4.38}$$

where, $R(t) = \displaystyle\int_0^t r(u)\mathrm{d}u$, and we use subscript iv to indicate the *idealized view*. The risk-averse formulation of the two-stage periodic PM optimization problem, under the *idealized view* can be represented as,

$$z^*(L)_{iv} = \min_{n \in \mathbb{N}_0} \left\{ \min_{T \in \left[0, \frac{L}{n}\right]} \bar{C}_\alpha(L; n, T)_{iv} \right\}, \tag{4.39}$$

where,

$$\bar{C}_\alpha(L; n, T)_{iv} = nC_{\mathrm{PM}} + \bar{\Lambda}_\alpha(L; n, T)_{iv}\ C_{\mathrm{CM}}, \tag{4.40}$$

$$\bar{\Lambda}_\alpha(L; n, T)_{iv} = \left(1 + \frac{\alpha}{2}\right) \Lambda(L; n, T)_{iv} + \frac{\alpha}{2} \Lambda_2(L; n, T)_{iv}, \tag{4.41}$$

and

$$\Lambda_2(L; n, T)_{iv} = \Lambda(L; n, T)_{iv} \left(\Lambda(L; n, T)_{iv} + 1\right). \tag{4.42}$$

Furthermore, our increasing failure rate assumption under the *idealized view*, leads to the following set of results, for the two-stage periodic PM optimization model.

Proposition 4.2.3.1 *If $r(\cdot)$ is increasing and continuously differentiable then for a fixed value of $n \in \mathbb{N}$,*

(i) $\Lambda(L;n,T)_{iv}$, $\Lambda_2(L;n,T)_{iv}$, $\bar{\Lambda}_\alpha(L;n,T)_{iv}$ and $\bar{C}_\alpha(L;n,T)_{iv}$ are convex in T on $\left[0, \frac{L}{n}\right]$, and

(ii) each of the $\Lambda(L;n,T)_{iv}$, $\Lambda_2(L;n,T)_{iv}$, $\bar{\Lambda}_\alpha(L;n,T)_{iv}$ and $\bar{C}_\alpha(L;n,T)_{iv}$ decreases on $\left[0, \frac{L}{n+1}\right]$, increases on $\left[\frac{L}{n+1}, \frac{L}{n}\right]$ and is minimized at $T = \frac{L}{n+1}$.

Therefore, increasing $r(\cdot)$ suffices to ensure that the optimal solution to the two-stage periodic PM optimization problem, under the idealized view, lies in $\mathbb{D} = \left\{d : d = \frac{L}{n}, n \in \mathbb{N}\right\}$.

Proof: (i) We recall that the sufficient condition for the convexity of $\Lambda\left(L;n,T\right)$ in T, for a fixed value of n and $T \in \left[0, \frac{L}{n}\right]$, in the two-stage model with imperfect PMs and minimal CMs, is given in equation (4.13). If we substitute $\breve{\theta}^{(j)}_{\text{PM}} = 0$, $j = 1, 2, \ldots, n$, in equation (4.13) then the sufficient condition reduces to $r'\left(T\right) \geq 0$, which holds since $r(\cdot)$ is increasing. Therefore, the increasing failure rate assumption under the *idealized view*, is sufficient for the convexity of $\Lambda(L;n,T)_{iv}$ leading to the convexity of $\Lambda_2(L;n,T)_{iv}$, $\bar{\Lambda}_\alpha(L;n,T)_{iv}$ and $\bar{C}_\alpha(L;n,T)_{iv}$ (see Proposition 4.1.2.2(i)).

(ii) From equation (4.38) we have,

$$\frac{\partial \Lambda(L;n,T)_{iv}}{\partial T} = n\left(r(T) - r(L - nT)\right) \begin{cases} < 0 & T \in \left[0, \frac{L}{n+1}\right) \\ = 0 & T = \frac{L}{n+1} \\ > 0 & T \in \left(\frac{L}{n+1}, \frac{L}{n}\right] \end{cases}, \tag{4.43}$$

which establishes that $\Lambda(L;n,T)_{iv}$ decreases on $\left[0, \frac{L}{n+1}\right]$, and increases on $\left[\frac{L}{n+1}, \frac{L}{n}\right]$. Part (i) of this proposition shows that under increasing $r(\cdot)$, $\Lambda(L;n,T)_{iv}$ is convex in T for a fixed n, and $T \in \left[0, \frac{L}{n}\right]$. Therefore, $\dfrac{\partial \Lambda(L;n,T)_{iv}}{\partial T} = 0$, is the necessary and sufficient condition that T minimizes $\Lambda(L;n,T)_{iv}$, which holds at $T = \frac{L}{n+1}$ (see equation (4.43)). We note that substituting $\breve{\theta}^{(j)}_{\mathrm{PM}} = 0$, $j = 1, 2, \ldots, n$, in equation (4.17), also establishes that $T = \frac{L}{n+1}$ minimizes $\Lambda(L;n,T)$, under the *idealized view*.

Similarly, we can show that, each of the $\Lambda_2(L;n,T)_{iv}$, $\bar{\Lambda}_\alpha(L;n,T)_{iv}$ and $\bar{C}_\alpha(L;n,T)_{iv}$ decreases on $\left[0, \frac{L}{n+1}\right]$, increases on $\left[\frac{L}{n+1}, \frac{L}{n}\right]$, and is minimized at $T = \frac{L}{n+1}$, by noting that

$$\frac{\partial \Lambda_2(L;n,T)_{iv}}{\partial T} \propto \frac{\partial \Lambda(L;n,T)_{iv}}{\partial T},$$

and

$$\frac{\partial \bar{C}_\alpha(L;n,T)_{iv}}{\partial T} = \frac{\partial \bar{\Lambda}_\alpha(L;n,T)_{iv}}{\partial T} \propto \frac{\partial \Lambda(L;n,T)_{iv}}{\partial T},$$

(from equations (4.42), (4.41) and (4.40)).

We have established that for a given $n \in \mathbb{N}_0$, $\bar{C}_\alpha(L;n,T)_{iv}$ is minimized at $T = \frac{L}{n+1}$, which suffices to show that the optimal solution to the two-stage periodic PM optimization problem, under the *idealized view*, lies in $\mathbb{D} = \left\{d : d = \frac{L}{n}, n \in \mathbb{N}\right\}$. $\square$

We next consider the single-stage periodic PM model, under the *idealized view*. We note that $\Lambda\left(L;T\right)$ defined in equation (4.25), for the single-stage model with imperfect PMs and minimal CMs, reduces to

$$\Lambda(L;T)_{iv} = \left\lfloor \frac{L}{T} \right\rfloor R\left(T\right) + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right), \tag{4.44}$$

in the *idealized view*. The risk-averse formulation of the single-stage periodic PM optimization problem, under the *idealized view*, becomes

$$z^*(L)_{iv} = \min_{T\in[0,L]} \bar{C}_\alpha(L;T)_{iv}, \tag{4.45}$$

where,

$$\bar{C}_\alpha(L;T)_{iv} = \left(\left\lceil\frac{L}{T}\right\rceil - 1\right) C_{\text{PM}} + \bar{\Lambda}_\alpha(L;T)_{iv}\, C_{\text{CM}}, \tag{4.46}$$

$$\bar{\Lambda}_\alpha(L;T)_{iv} = \left(1+\frac{\alpha}{2}\right)\Lambda(L;T)_{iv} + \frac{\alpha}{2}\Lambda_2(L;T)_{iv}, \tag{4.47}$$

and

$$\Lambda_2(L;T)_{iv} = \Lambda(L;T)_{iv}\left(\Lambda(L;T)_{iv}+1\right). \tag{4.48}$$

Furthermore, our increasing failure rate assumption under the idealized view, leads to the following set of results.

Proposition 4.2.3.2 *If $r(\cdot)$ is increasing then,*

(i) *(a)* *$\Lambda(L;T)_{iv}$, $\Lambda_2(L;T)_{iv}$ and $\bar{\Lambda}_\alpha(L;T)_{iv}$ are continuous in T.*

(b) *$\bar{C}_\alpha(L;T)_{iv}$ is lower semicontinuous in T with the set of discontinuities, $\mathbb{D} = \left\{d : d = \frac{L}{n}, n \in \mathbb{N}\right\}$.*

(ii) *(a)* *$\Lambda(L;T)_{iv}$, $\Lambda_2(L;T)_{iv}$ and $\bar{\Lambda}_\alpha(L;T)_{iv}$ are increasing and convex in T for $n \in \mathbb{N}$ and $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$.*

(b) *$\bar{C}_\alpha(L;T)_{iv}$ is increasing and convex in T for $n \in \mathbb{N}$ and $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$.*

(iii) *(a)* *an optimal solution to the single-stage periodic PM optimization problem under the idealized view lies in $\mathbb{D} = \left\{d : d = \frac{L}{n}, n \in \mathbb{N}\right\}$.*

(b) $\displaystyle\min_{n\in\mathbb{N}_0}\left\{\min_{T\in\left[0,\frac{L}{n}\right]} \bar{C}_\alpha(L;n,T)_{iv}\right\} = \min_{T\in[0,L]} \bar{C}_\alpha(L;T)_{iv}$

Proof: *(i)* We recall that maintenance models under the *idealized view* are special cases of the imperfect PM models, and therefore the continuity results from

Proposition 4.2.2.1 apply to $\Lambda(L;T)_{iv}$, $\Lambda_2(L;T)_{iv}$, $\bar{\Lambda}_\alpha(L;T)_{iv}$ and $\bar{C}_\alpha(L;T)_{iv}$.

(ii) Proposition 4.2.3.1 establishes that each of $\Lambda(L;n,T)_{iv}$, $\Lambda_2(L;n,T)_{iv}$, $\bar{\Lambda}_\alpha(L;n,T)_{iv}$ and $\bar{C}_\alpha(L;n,T)_{iv}$ is increasing and convex in T, for a fixed n and $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$, under the increasing failure rate assumption, in the *idealized view*. Since *idealized view* is a special case of imperfect PMs and minimal CMs, proof of this part is complete by noting the following results from Proposition 4.2.2.2 (see parts (i) – (iii)):

(a) $\Lambda(L;n,T)_{iv} = \Lambda(L;T)_{iv}$, $\Lambda_2(L;n,T)_{iv} = \Lambda_2(L;T)_{iv}$, and $\bar{\Lambda}_\alpha(L;n,T)_{iv} = \bar{\Lambda}_\alpha(L;T)_{iv}$ for $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right]$,

(b) $\bar{C}_\alpha(L;n,T)_{iv} = \bar{C}_\alpha(L;T)_{iv}$ for $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$.

(iii) (a) From parts (i) and (ii) of this proposition, we note that $\bar{C}_\alpha(L;T)_{iv}$ is lower semicontinuous in T with the set of discontinuities, $\mathbb{D} = \left\{d : d = \frac{L}{n}, n \in \mathbb{N}\right\}$, and increasing in T, for $T \in \left[\frac{L}{n+1}, \frac{L}{n}\right)$, and a fixed $n \in \mathbb{N}$. Therefore, $\bar{C}_\alpha(L;T)_{iv}$ is minimized at one of the points in $\mathbb{D}$.

(iii) (b) We show this part by contradiction. Let n^* and T^* minimize the two-stage periodic PM problem, under the *idealized view*. From Proposition 4.2.3.1 and the proof of part (ii) of this proposition,

$$T^* = \frac{L}{n^*+1}$$
$$\bar{C}_\alpha\left(L;n^*,T^*\right)_{iv} = \bar{C}_\alpha\left(L;T^*\right)_{iv}. \tag{4.49}$$

Let $\hat{T}^*$ be the optimal solution to the single-stage periodic PM problem under the *idealized view* and assume

$$\bar{C}_\alpha\left(L;T^*\right)_{iv} > \bar{C}_\alpha\left(L;\hat{T}^*\right)_{iv}. \tag{4.50}$$

Since $\hat{T}^* \in \mathbb{D}$ (see part (*iii*) (*a*) of this proposition), there exists $\hat{n}^* \in \mathbb{N}_0$ such that $\hat{T}^* = \frac{L}{\hat{n}^*+1}$. Once again, from the proof of part (*ii*) of this proposition, we have

$$\bar{C}_\alpha \left(L; \hat{T}^*\right)_{iv} = \bar{C}_\alpha \left(L; \hat{n}^*, \hat{T}^*\right)_{iv}. \tag{4.51}$$

From equations (4.49), (4.50) and (4.51),

$$\bar{C}_\alpha \left(L; n^*, T^*\right)_{iv} > \bar{C}_\alpha \left(L; \hat{n}^*, \hat{T}^*\right)_{iv},$$

which contradicts that n^* and T^* solve the two-stage periodic PM problem, under the *idealized view*. Hence,

$$\min_{n \in \mathbb{N}_0} \left\{ \min_{T \in \left[0, \frac{L}{n}\right]} \bar{C}_\alpha(L; n, T)_{iv} \right\} = \min_{T \in [0, L]} \bar{C}_\alpha(L; T)_{iv}. \qquad \square$$

We note that Galenko et al. (2005) establish parts (*i*), (*ii*), and (*iii*)(*a*) of Proposition 4.2.3.2, for the *risk-neutral* version of the single-stage periodic PM problem. We show that the results extend to the corresponding *risk-averse* formulation. In addition, we prove that the optimal solutions to the single-stage and the two-stage periodic PM optimization models are equal and lie in $\mathbb{D}$. Therefore, the single-stage/two-stage periodic PM problem under the *idealized view* can be solved by comparing $\bar{C}_\alpha(L; T)_{iv}/\bar{C}_\alpha(L; n, T)_{iv}$, at $T = \frac{L}{n+1}/\left(n, T = \frac{L}{n+1}\right)$, for $n = 0, 1, \ldots, n_{\max}$, where $n_{\max}$ represents the maximum number of PMs that can possibly be scheduled over the finite planning horizon, $[0, L]$. Furthermore, we recall that under the *idealized view*, Galenko et al. (2005) present an efficient algorithm to solve the *risk-neutral* version of the single-stage periodic PM model. In the following, we extend the efficient algorithm to the *risk-averse* formulation of the single-stage (and equivalently two-stage) periodic PM problem, under the *idealized* *view*. Towards this, we substitute $\left\lfloor \frac{L}{T} \right\rfloor$ with $\frac{L}{T}$ in equation (4.44), and write

$$\Lambda_r(L;T)_{iv} = \frac{L}{T}R\left(T\right) + R\left(L - \frac{L}{T}T\right) = \frac{L}{T}R\left(T\right). \tag{4.52}$$

We next consider a *relaxed* version of the single-stage periodic PM model under the *idealized view*:

$$z_r^*(L)_{iv} = \min_{T\in[0,L]} \bar{C}_{\alpha_r}(L;T)_{iv}, \tag{4.53}$$

where,

$$\bar{C}_{\alpha_r}(L;T)_{iv} = \left(\frac{L}{T} - 1\right) C_{\text{PM}} + \bar{\Lambda}_{\alpha_r}(L;T)_{iv}\ C_{\text{CM}}, \tag{4.54}$$

$$\bar{\Lambda}_{\alpha_r}(L;T)_{iv} = \left(1 + \frac{\alpha}{2}\right) \Lambda_r(L;T)_{iv} + \frac{\alpha}{2}\Lambda_{2_r}(L;T)_{iv}, \tag{4.55}$$

and

$$\Lambda_{2_r}(L;T)_{iv} = \Lambda_r(L;T)_{iv}\left(\Lambda_r(L;T)_{iv} + 1\right). \tag{4.56}$$

In the following proposition, we characterize $\bar{C}_{\alpha_r}(L;T)_{iv}$, establish its relationship with $\bar{C}_{\alpha}(L;T)_{iv}$, and present a key result that helps to solve the single-stage periodic PM model efficiently.

Proposition 4.2.3.3 *If $r(\cdot)$ is increasing then,*

(*i*) *$\bar{C}_{\alpha_r}(L;T)_{iv}$ is quasiconvex in T for $T \in [0, L]$.*

(*ii*) *$\bar{C}_{\alpha_r}(L;T)_{iv} = \bar{C}_{\alpha}(L;T)_{iv}$ for $T \in \mathbb{D}$.*

(*iii*) *Furthermore, if T_r^* and T^* minimize $\bar{C}_{\alpha_r}(L;T)_{iv}$ and $\bar{C}_{\alpha}(L;T)_{iv}$, respectively, then $T^* \in \arg\min_{T\in\left\{\frac{L}{n^*+1}, \frac{L}{n^*}\right\}} \bar{C}_{\alpha_r}(L;T)_{iv}$, where $n^* \in \mathbb{N}_0$ satisfies $\frac{L}{n^*+1} \leq T_r^* < \frac{L}{n^*}$.*

Proof: (*i*) Let S_γ be the lower-level set of $\bar{C}_{\alpha_r}(L;T)_{iv}$:

$$S_\gamma = \left\{T : \bar{C}_{\alpha_r}(L;T)_{iv} \leq \gamma\right\}. \tag{4.57}$$

Now, $\bar{C}_{\alpha_r}(L;T)_{iv}$ is quasiconvex in T if S_γ is a convex set. From equations (4.54), (4.55), (4.56) and (4.52), we have

$$\bar{C}_{\alpha_r}(L;T)_{iv} = \left(\frac{L}{T} - 1\right) C_{\mathrm{PM}} + (1+\alpha)\ \frac{L}{T}\ R(T) + \frac{\alpha}{2}\ \frac{L}{T}\ \left(R(T)\right)^2. \tag{4.58}$$

Substituting $\bar{C}_{\alpha_r}(L;T)_{iv}$ from equation (4.58) in equation (4.57), we obtain

$$S_\gamma = \left\{T : L\ C_{\mathrm{PM}} + (1+\alpha)\ L\ R(T) + \frac{\alpha}{2}\ L\ \left(R(T)\right)^2 \leq T\left(\gamma + C_{\mathrm{PM}}\right)\right\}. \tag{4.59}$$

Since $r(\cdot)$ is increasing, $R(\cdot)$ and hence $(R(\cdot))^2$ are convex, and therefore S_γ is convex.

(ii) We note that $T \in \mathbb{D}$ implies that $T = \frac{L}{n}$, for some $n \in \mathbb{N}$. Therefore, $\frac{L}{T} = \left\lceil \frac{L}{T} \right\rceil = \left\lfloor \frac{L}{T} \right\rfloor = n$, and from equations (4.44) and (4.52) we have,

$$\Lambda(L;T)_{iv}\big|_{T=\frac{L}{n}} = \left\lfloor \frac{L}{T} \right\rfloor R\left(T\right) + R\left(L - \left\lfloor \frac{L}{T} \right\rfloor T\right) = nR(T) = \Lambda_r(L;T)_{iv}\big|_{T=\frac{L}{n}}. \tag{4.60}$$

Furthermore, equations (4.47), (4.48), (4.55), (4.56), and (4.60) imply,

$$\bar{\Lambda}_\alpha(L;T)_{iv}\big|_{T=\frac{L}{n}} = \bar{\Lambda}_{\alpha_r}(L;T)_{iv}\big|_{T=\frac{L}{n}}. \tag{4.61}$$

Finally, from equations (4.46), (4.54) and (4.61), we obtain

$$\begin{aligned}
\bar{C}_\alpha(L;T)_{iv}\big|_{T=\frac{L}{n}} &= \left(\left\lceil \frac{L}{T} \right\rceil - 1\right) C_{\mathrm{PM}} + \bar{\Lambda}_\alpha(L;T)_{iv}\big|_{T=\frac{L}{n}}\ C_{\mathrm{CM}} \\
&= (n-1)C_{\mathrm{PM}} + \bar{\Lambda}_{\alpha_r}(L;T)_{iv}\big|_{T=\frac{L}{n}}\ C_{\mathrm{CM}} \\
&= \bar{C}_{\alpha_r}(L;T)_{iv}\big|_{T=\frac{L}{n}}.
\end{aligned}$$

(iii) If T^* minimizes $\bar{C}_\alpha(L;T)_{iv}$ then

$$\bar{C}_\alpha(L;T^*)_{iv} = \min_{T\in[0,L]} \bar{C}_\alpha(L;T)_{iv}. \tag{4.62}$$

From part $(iii)(a)$ of Proposition 4.2.3.2 and part (ii) this proposition, we have

$$\min_{T\in[0,L]} \bar{C}_\alpha(L;T)_{iv} = \min_{T\in\mathbb{D}} \bar{C}_\alpha(L;T)_{iv} = \min_{T\in\mathbb{D}} \bar{C}_{\alpha_r}(L;T)_{iv}. \tag{4.63}$$

Furthermore, part (i) of this proposition establishes that $\bar{C}_{\alpha_r}(L;T)_{iv}$ is quasiconvex in T. Therefore, if T_r^* minimizes $\bar{C}_{\alpha_r}(L;T)_{iv}$, then $\bar{C}_{\alpha_r}(L;T)_{iv}$ is nonincreasing on $[0,T_r^*]$ and nondecreasing on $[T_r^*,L]$. Thus, within set $\mathbb{D}$, $\bar{C}_{\alpha_r}(L;T)_{iv}$ is minimized at one of the two points given by the set $\left\{\frac{L}{n^*+1}, \frac{L}{n^*}\right\}$ such that, $\frac{L}{n^*+1} \leq T_r^* < \frac{L}{n^*}$ for some $n^* \in \mathbb{N}_0$ and hence,

$$\min_{T\in\mathbb{D}} \bar{C}_{\alpha_r}(L;T)_{iv} = \min_{T\in\left\{\frac{L}{n^*+1}, \frac{L}{n^*}\right\}} \bar{C}_{\alpha_r}(L;T)_{iv}. \tag{4.64}$$

From equations (4.62), (4.63) and (4.64), we have

$$\bar{C}_\alpha(L;T^*)_{iv} = \min_{T\in\left\{\frac{L}{n^*+1}, \frac{L}{n^*}\right\}} \bar{C}_{\alpha_r}(L;T)_{iv},$$

i.e.,

$$T^* = \text{argmin}_{T\in\left\{\frac{L}{n^*+1}, \frac{L}{n^*}\right\}} \bar{C}_{\alpha_r}(L;T)_{iv}. \qquad \square$$

Proposition 4.2.3.3 establishes that $\bar{C}_{\alpha_r}(L;T)_{iv}$ is quasiconvex in T. Therefore, we can efficiently find $T_r^* \in \arg\min_{T\in[0,L]} \bar{C}_{\alpha_r}(L;T)_{iv}$, using well known line search methods, such as, a golden section search or Fibonacci search (see, e.g., Luenberger 1984). Once we have obtained T_r^*, we identify two points in $\mathbb{D}$, first to the immediate left, and second to the immediate right of T_r^*. Proposition 4.2.3.3 shows that an optimal solution to the single-stage (or equivalently two-stage) periodic PM optimization problem, T^*, under the *idealized view*, is one of the two identified points in $\mathbb{D}$, that are closest to T_r^*.

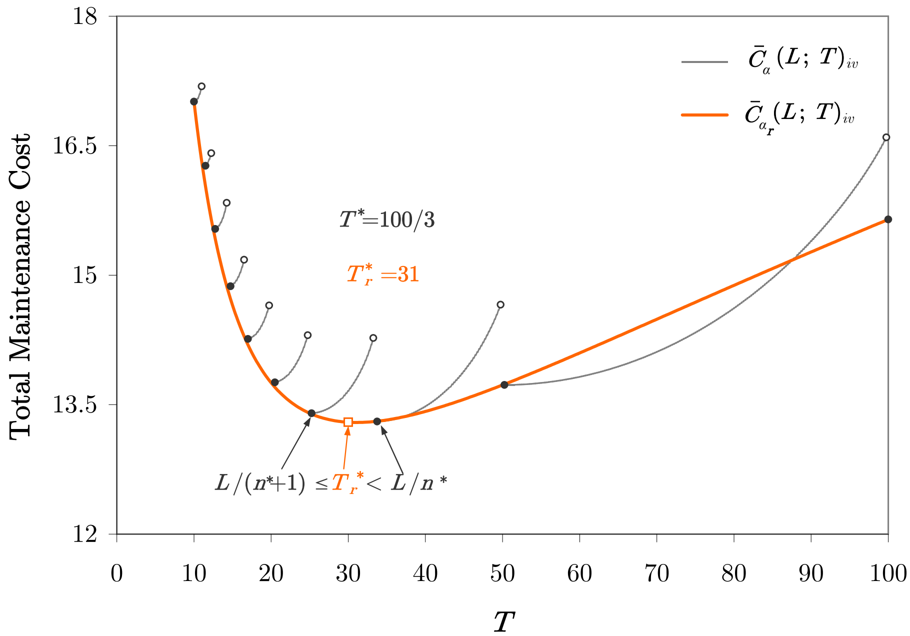

Figure 4.2: Relationship between $\bar{C}_{\alpha_r}(L;T)_{iv}$ and $\bar{C}_{\alpha}(L;T)_{iv}$, and T_r^* and T^*, where $T_r^* = \arg\min_{T\in[0,L]}\bar{C}_{\alpha_r}(L;T)_{iv}$, and $T^* = \arg\min_{T\in[0,L]}\bar{C}_{\alpha}(L;T)_{iv}$.

Figure 4.2 summarizes the results from Proposition 4.2.3.3. We re-consider the set of parameters used in Figure 4.1 from Section 4.2.2, under the *idealized view*. We note that the two objective functions, $\bar{C}_{\alpha_r}(L;T)_{iv}$ and $\bar{C}_{\alpha}(L;T)_{iv}$ are equal, when $T \in \mathbb{D}$. Furthermore, $\frac{100}{4} < T_r^* = 31 < \frac{100}{3}$, and therefore, T^*, is one of the two points $\{\frac{100}{4}, \frac{100}{3}\}$, which minimizes $\bar{C}_{\alpha_r}(L;T)_{iv}$ (or equivalently $\bar{C}_{\alpha}(L;T)_{iv}$). For the set of parameters considered, $T^* = \frac{100}{3}$.

Chapter 5

Conclusion

In this dissertation, we have developed analytical and simulation-based periodic preventive maintenance (PM) optimization models, for a single piece of equipment. The equipment under consideration, deteriorate with time or use, and can be repaired upon failure, through corrective maintenance (CM). The effectiveness of the two types of maintenance actions (PM and CM), is modeled through age reduction factors. In the simulation-based models, we assume that both types of maintenance actions are imperfect, whereas our analytical models consider imperfect PMs with minimal CMs. In our optimization models, we seek an optimal periodic PM policy that minimizes the sum of the expected total cost of PMs, and the risk-averse cost of CMs, over a finite planning horizon. The models developed in this research are motivated by the maintenance problems arising at South Texas Project Nuclear Operating Company (STPNOC), but can be applied to general production systems of repairable equipment.

Chapter 2 describes a Bayesian approach to parameter estimation, in which, prior knowledge of the system operators, and the likelihood of observing the data, yield posterior estimates for the failure rate and the maintenance effectiveness parameters. Our preference to use Bayesian methods over classical parameter estimation procedures, is primarily due to the small and heavily censored datasets from STPNOC, to which classical methods are numerically sensitive. We suggest general shapes of beta priors, for the two maintenance effectiveness parameters, and describe a random-walk-based Gibbs sampler. We provide posterior estimates for three datasets, including a dataset from STPNOC. Our posterior estimates for the two datasets from the literature are consistent with published results. Furthermore, our computational results successfully demonstrate that our Gibbs sampler is arguably the obvious choice over rejection sampling-based Bayesian parameter estimation, for this class of problems.

Chapter 3 is motivated by the need to simulate the equipment failure process to estimate the expected number of failures, when its closed-form expression is not available. In fact, such closed-form expressions are only available, when CMs are modeled as minimal repairs, under the well-known *Kijima-II* age-reduction model. A general age-reduction model may also necessitate a simulation-based approach. We extend the simulation procedure known in the literature to estimate the expected number of failures, from the estimated joint posterior obtained in Chapter 2. This leads to a versatile simulation-based periodic PM optimization model. Optimal periodic PM policies, under classical maximum likelihood (ML) and Bayesian estimates are obtained for all the three datasets. Limitations of the ML approach are revealed for a dataset from the literature, in which the use of ML estimates of the parameters, in the maintenance optimization model, under the *idealized view*, fails to capture a *trivial* optimal PM policy.

In Chapter 4, we introduce a general two-stage formulation of the risk-averse periodic PM optimization model, with imperfect PMs and minimal CMs. For general values of PM age reduction factors, we provide sufficient conditions to establish the convexity of the first and second moments of the number of failures, and the risk-averse expected total maintenance cost, over a finite planning horizon. In one of our significant results, we show that these convexity results are independent of the PM age reduction factors for increasing Weibull rates and a general class of increasing and convex failure rates. We extend all of our convexity results from the two-stage model to the single-stage periodic PM model. In general, the optimal periodic PM policy under the single-stage model is no better than the optimal two-stage policy. But if PMs are assumed perfect, then we establish that the single-stage and the two-stage optimization models are equivalent.

5.1 Summary of Contributions

The main contributions of this dissertation lie in the development of statistical, simulation, and optimization models that aim at improving the safety and financial performance measures of nuclear power plants. In the following, we summarize our contributions to a broad area that integrates operations research, reliability and statistics:

- A study of virtual-age-based, age-reduction models: The inter-failure times of a repairable piece of equipment are often dependent. Therefore, well-established statistical tools for the analysis of i.i.d. observations rarely apply to the maintenance data of a repairable unit of equipment. This research demonstrates that the virtual-age-based, age-reduction models, provide a general framework for modeling and analysis of the maintenance data from a repairable piece of equipment, in terms of the failure rate of the time to its first failure.

- A new adaptation of a random-walk-based Gibbs sampler for virtual-age-based, age-reduction models: To our knowledge, we are the first to apply a well-adapted variant of a Gibbs sampler to estimate the failure rate and the maintenance effectiveness parameters within a virtual-age-based, age-reduction framework.

- A new simulation-based periodic PM optimization model: We extend the simulation procedure known in the literature to estimate the expected number of failures in the Bayesian setting, which leads to a new simulation-based periodic PM optimization model, under a finite planning horizon.

- Two new risk-averse periodic PM optimization models and their convexity results: We introduce a risk-averse CM cost function, which is best suited for STPNOC, in particular, and the US nuclear industry, in general. We also introduce finite horizon, risk-averse formulations of the single-stage and two-stage periodic PM optimization model, with imperfect PMs and minimal CMs, and establish new convexity results for both models.

5.2 Directions for Future Research

Our convexity results for the two periodic PM optimization model assume that the PM age reduction factors are known in advance. A natural extension of this work is to see if the convexity results still apply, when the PM age reduction factors are unknown, and can be modeled as random variables. Another extension of this work is to develop both analytical and simulation-based *sequential* PM optimization models, in the risk-averse setting. In this research, we assume that uncertainty in PM quality can only be estimated a priori. But in certain situations, it is possible to estimate the quality of a PM after it has been performed. In these situations, the *sequential* PM optimization model can be extended to a multi-stage stochastic optimization problem.

Bibliography

Albert, J. (2009), *Bayesian Computation with R (Use R)*, 2nd edn, Springer, New York, NY, USA.

Anderson, P. K., Borgan, O., Gill, R. D. & Keiding, N. (1993), *Statistical Methods based on Counting Processes*, Springer-Verlag, New York, NY, USA.

Assaf, D. & Shanthikumar, J. G. (1987), 'Optimal group maintenance policies with continuous and periodic inspections', *Management Science* **33**, 1440 – 1452.

Baker, R. D. (1991), 'Testing the efficacy of preventive maintenance', *Journal of the Operational Research Society* **42**(6), 493 – 503.

Barlow, R. E. & Hunter, L. C. (1960), 'Optimum preventive maintenance policies', *Operations Research* **8**(1), 127 – 146.

Beichelt, F. & Fischer, K. (1980), 'General failure model applied to preventive maintenance policies', *IEEE Transactions of Reliability* **29**(1), 39 – 41.

Bloch, H. & Geitner, F. (1994), *Practical Machinery Management for Process Plants, Machinery Failure Analysis and Troubleshooting, Vol. 2*, Gulf Publishing, Houston, TX, USA.

Boland, P. J. & Proschan, F. (1982), 'Periodic replacement with increasing minimal repair costs at failure', *Operations Research* **30**(6), 1183 – 1189.

Casella, G. & Berger, R. L. (2002), *Statistical Inference*, 2nd edn, Duxbury, Thomson Learning, Pacific Grove, CA, USA.

Cho, I. D. & Parlar, M. (1991), 'A survey of maintenance models for multi-unit systems', *European Journal of Operational Research* **51**, 1 – 23.

Chun, Y. H. (1992), 'Optimal number of periodic preventive maintenance operations under warranty', *Reliability Engineering and System Safety* **37**, 223 – 225.

Dekker, R., Wildeman, R. E. & Van Der Duyn Schouten, F. A. (1997), 'A review of multi-component maintenance models with economic dependence', *Mathematical Methods of Operational Research* **45**(3), 411 – 435.

Galenko, A., Morton, D., Popova, E., Kee, E., Grantom, R. & Sun, A. (2005), Operational level models and methods for risk informed nuclear asset management, *in* 'Proceedings of the American Nuclear Society International Topical Meeting on Probabilistic Safety Assessment'.

Haurie, A. & L'Ecuyer, P. (1982), 'A stochastic control approach to group preventive replacement in a multicomponent system', *IEEE Transactions on Automatic Control* **27**, 387 – 393.

Jack, N. (1997), 'Analysing event data from a repairable machine subject to imperfect preventive maintenance', *Quality and Reliability Engineering International* **13**, 183 – 186.

Jack, N. (1998), 'Age-reduction models for imperfect maintenance', *IMA Journal of Mathematics Applied in Business and Industry* **9**(4), 347 – 354.

Kijima, M. (1989), 'Some results for repairable systems with general repair', *Journal of Applied Probability* **26**(1), 89 – 102.

Kijima, M., Morimura, H. & Suzuki, Y. (1988), 'Periodical replacement problem without assuming minimal repair', *European Journal of Operational Research* **37**(2), 194 – 203.

Lai, M.-T. & Chen, Y.-C. (2006), 'Optimal periodic replacement policy for a two-unit system with failure rate interaction', *The International Journal of Advanced Manufacturing Technology* **29**, 367 – 371.

Luenberger, D. G. (1984), *Linear and Non-Linear Programming*, 2nd edn, Addison-Wesley Inc., Reading, MA, USA.

McCall, J. J. (1965), 'Maintenance policies for stochastically failing equipment: A survey', *Management Science* **11**, 493 – 524.

McGoldrick, F. (2011), Limiting transfers of enrichment and reprocessing technology: Issues, constraints, options, Technical report, Belfer Center for Science and International Affairs, Harvard Kennedy School, Cambridge, MA.

Murthy, D. N. P. & Nguyen, D. G. (1985*a*), 'Study of a multi-component system with failure interaction', *European Journal of Operational Research* **21**, 330 – 338.

Murthy, D. N. P. & Nguyen, D. G. (1985*b*), 'A study of two-component system with failure interaction', *Naval Research Logistics Quarterly* **32**, 239 – 247.

Muth, E. J. (1977), 'An optimal decision rule for repair vs replacement', *IEEE Transactions of Reliability* **26**(3), 179 – 181.

Nakagawa, T. (1979), 'Optimum policies when preventive maintenance is imperfect', *IEEE Transactions of Reliability* **28**(4), 331 – 332.

Nakagawa, T. (1988), 'Sequential imperfect preventive maintenance policies', *IEEE Transactions of Reliability* **37**(3), 295 – 298.

Nakagawa, T. & Mizutani, S. (2009), 'A summary of maintenance policies for a finite interval', *Reliability Engineering and System Safety* **94**(1), 89 – 96.

Nguyen, D. G. & Murthy, D. N. P. (1981), 'Optimal preventive maintenance policies for repairable systems', *Operations Research* **29**(6), 1181 – 1194.

Nicolai, R. P. & Dekker, R. (2006), Optimal maintenance of multi-component systems: A review, Technical Report EI 2006-29, Econometric Institute, Erasmus University Rotterdam, The Netherlands.

NRC (2011), 'Expected new nuclear power plant applications in the USA', `http://www.nrc.gov/reactors/new-reactors/new-licensing-files/expected-new-rx-applications.pdf`.

Okumoto, K. & Elsayed, E. A. (1983), 'An optimum group maintenance policy', *Naval Research Logistics Quarterly* **30**, 667 – 674.

Park, K. S. (1979), 'Optimal number of minimal repairs before replacement', *IEEE Transactions of Reliability* **28**(2), 137 – 140.

Pham, H. & Wang, H. (1996), 'Imperfect maintenance', *European Journal of Operational Research* **94**(3), 425 – 438.

Pierskalla, W. P. & Voelker, J. A. (1979), 'A survey of maintenance models: The control and surveillance of deteriorating systems', *Naval Research Logistics Quarterly* **23**, 353 – 388.

Popova, E., Morton, D. & Damien, P. (2005), Risk informed nuclear asset management - models and methods, Technical report, NSF.

Popova, E. & Wilson, J. (1999), 'Group replacement policies for parallel systems whose components have phase distributed failure times', *Annals of Operations Research* **91**, 163 – 189.

Rausand, M. & Høyland, A. (2004), *System Reliability Theory: Models, Statistical Methods and Applications*, 2nd edn, John Wiley and Sons, New York, NY, USA.

Ritchken, P. & Wilson, J. G. (1990), '(m, t) group maintenance policies', *Management Science* **36**, 632 – 639.

Robert, C. P. & Casella, G. (2005), *Monte Carlo Statistical Methods (Springer Texts in Statistics)*, Springer-Verlag, Secaucus, NJ, USA.

Thomas, L. C. (1986), 'A survey of maintenance and replacement models for maintainability and reliability of multi-item systems', *Reliability Engineering* **16**(4), 297 – 309.

Thompson, W. A. (1981), 'On the foundations of reliability', *Technometrics* **23**(1), 1 – 13.

Valdez-Flores, C. & Feldman, R. (1989), 'A survey of preventive maintenance models for stochastically deteriorating single-unit systems', *Naval Research Logistics Quarterly* **39**, 419 – 446.

Wang, H. (2002), 'A survey of maintenance policies of deteriorating systems', *European Journal of Operational Research* **139**(3), 469 – 489.

WNA (2011), 'World nuclear power reactors and uranium requirements', `http://www.world-nuclear.org/info/reactors.html`.

Yu, P., Song, J. J. & Cassady, C. R. (2008), Parameter estimation for a repairable system under imperfect maintenance, *in* 'Proceedings of the 2008 Annual Reliability and Maintainability Symposium', IEEE Computer Society, Washington, DC, USA, pp. 428 – 433.

Vita

Inderjeet Singh, the youngest son of Satnam Kaur and Gurdeep Singh, was born in Ludhiana (Punjab), India, on November 12, 1978. He attended Chanan Devi Memorial Public School (until 8th standard), Malwa Khalsa Senior Secondary School (9th and 10th standards) and Satish Chander Dhawan Government College for Boys (11th and 12th standards), in his home town Ludhiana. He received his B. Tech. in Industrial Engineering from Dr. B. R. Ambedkar National Institute of Technology (formerly Regional Engineering College), Jalandhar, in July 1999, graduating with honors. He started his career as a visiting lecturer in Government Polytechnic for Women, Ludhiana, in September 1999. In February 2000, he joined his alma mater as a full time lecturer in the department of Industrial Engineering. In July 2001, he secured admission in the Indian Institute of Technology, Bombay, and finished his M. Tech. in Industrial Engineering & Operations Research in January 2003. He worked as a scientist in the Vikram Sarabhai Space Center of the Indian Space Research Organization in Thiruvananthapuram from March 2003 to July 2004.

In August 2004, he began his doctoral studies in the Operations Research & Industrial Engineering program at the University of Texas at Austin. During his time at UT, he worked as a teaching assistant in the department of Human Ecology, and as a research assistant to Dr. Elmira Popova in Operations Research & Industrial Engineering. In addition, he completed three summer internships, two at Dell Inc. in Austin, TX, and one at the South Texas Project Nuclear Operating Company in Wadsworth, TX. He is the recipient of the 2010 Bruton fellowship from the University of Texas at Austin. He completed his doctoral research on risk-averse periodic preventive maintenance optimization with Dr. Elmira Popova and Dr. David P. Morton in July 2011, and joined the science group at Predictix LLC in Atlanta, GA as a retail science consultant, in August 2011. He is currently a member of the Institute for Operations Research and the Management Sciences (INFORMS).

Permanent Address: S/o. S. Gurdeep Singh Chawla
H.No. 342/1-K, Near Doaba Workshop
Netaji Nagar, Saleem Tabri
Ludhiana (Punjab) – 141008
India

This dissertation was typeset with LaTeX 2_ε[1] by the author.

[1] LaTeX 2_ε is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The macros used in formatting this dissertation were written by Dinesh Das, Department of Computer Sciences, The University of Texas at Austin, and extended by Bert Kay, James A. Bednar, and Ayman El-Khashab.